Resonant mode confinement and lifetime in equilateral triangular dielectric cavities

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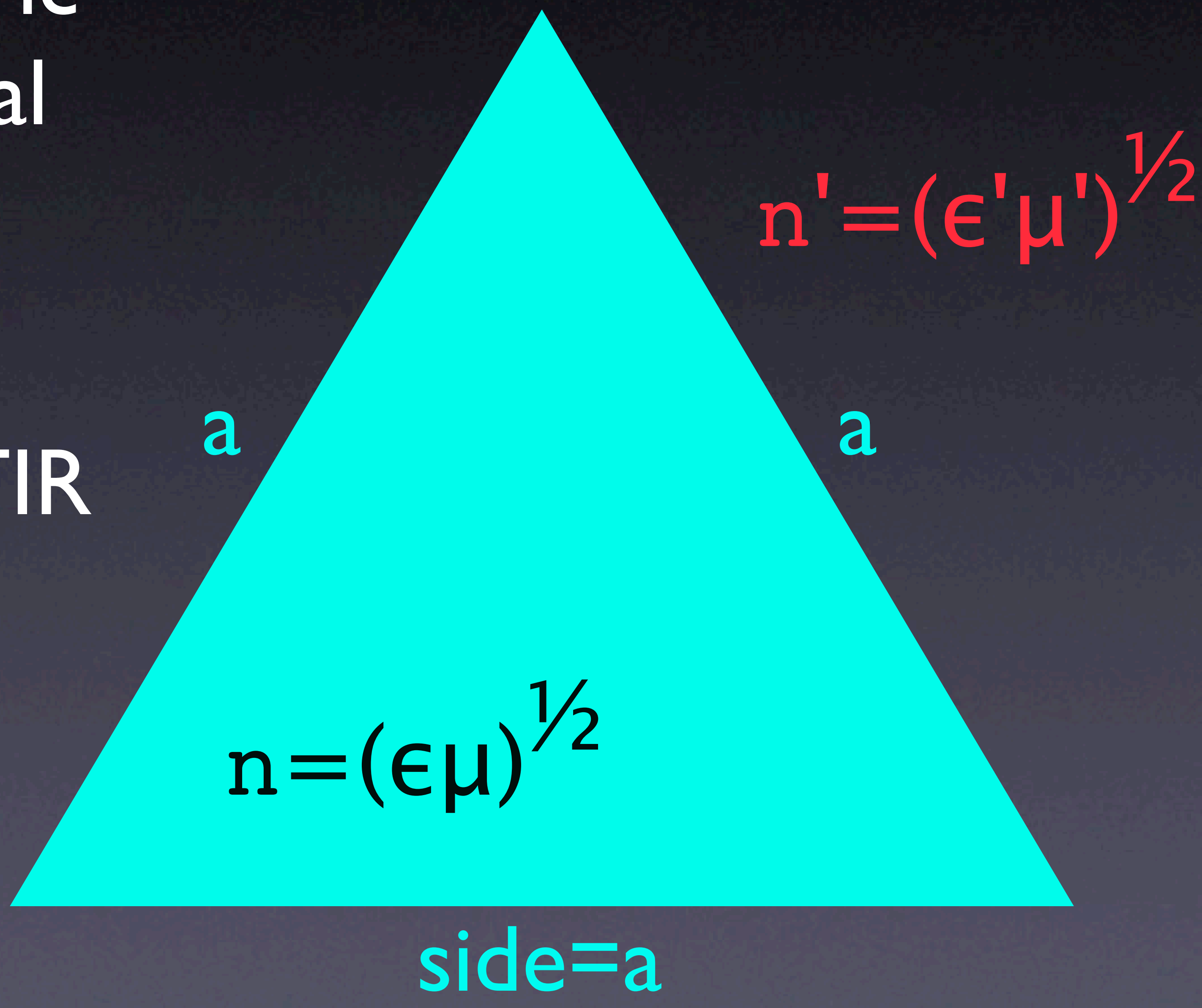
An interesting problem in theoretical and applied physics

- high index dielectric cavity of equilateral triangular cross section, height h
- what are the 2D TIR resonant modes?
- what are their lifetimes or Q-factors?

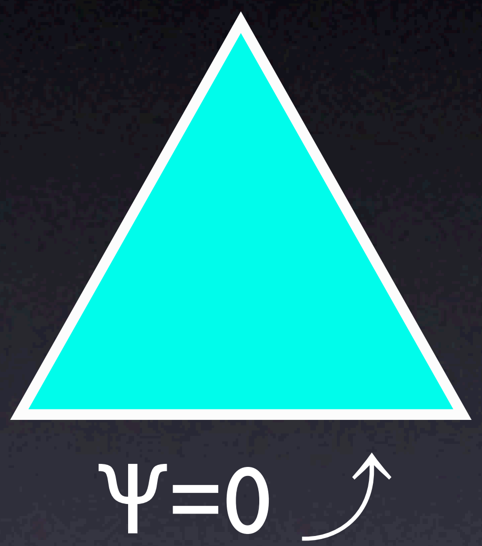


Motivation for studies

- mode selection in micro-cavity resonators
- advances in growth of semiconductor geometries -- circles, triangles, squares, hexagons, pyramids, etc., experimental data
- challenging and fun problem in electromagnetism, field matching conditions, polarization dependence
- relation to quantum polygonal billiards

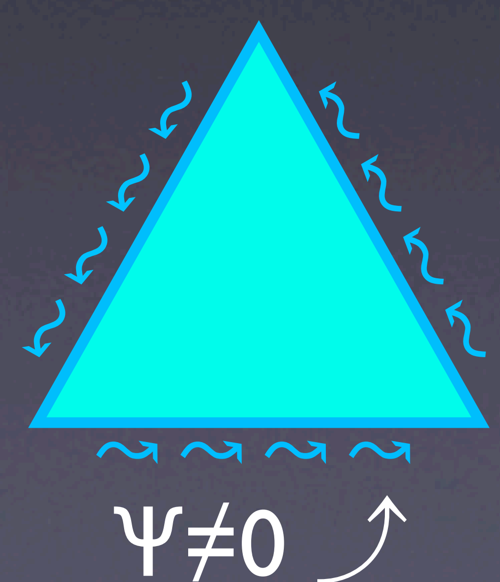
Some previous works on ETRs

- M. G. Lamé (1852). Analytic soln. for elastic waves on a triangular drum, Helmholtz eq. with Ψ=0 on boundary (scalar Ψ, Dirichlet Boundary Conditions)



- H. C. Chang et al. (2000). Used Lamé soln. for TM modes in dielectric ETR, applying DBC.

- Y. Z. Huang et al. (1999-2001). Approx. soln. for TM and TE modes using Maxwell eqs. and their boundary conditions in a dieletric ETR. Evanescent exterior waves.



My 'simplified analysis', various assumptions

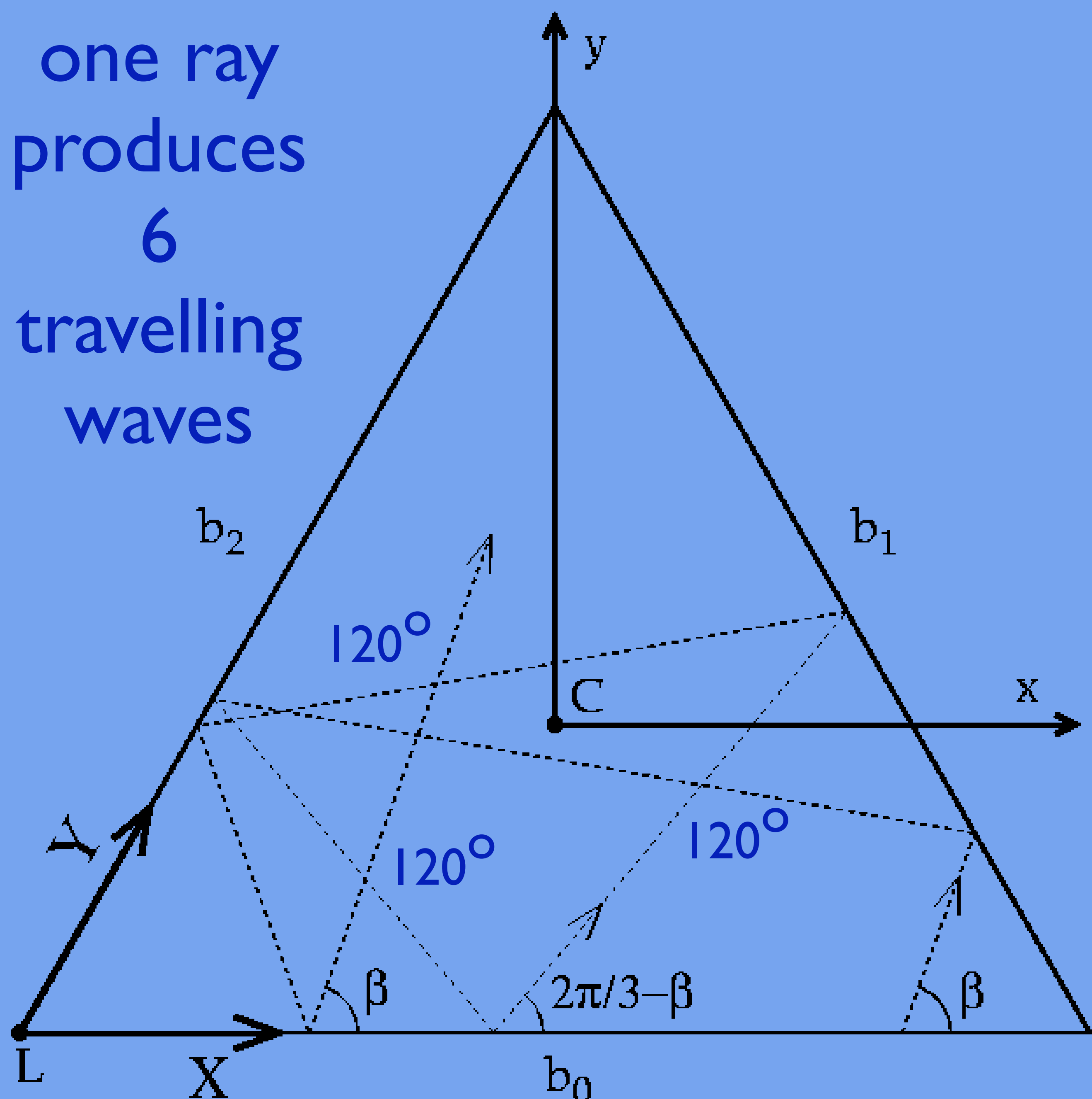
- Large index ratio $N \equiv n/n' \gg 1$,
  $\Rightarrow$ modes' fields strongly confined by TIR.
- No z-dependence; independent TE and TM modes.
- Initial approx: use DBC on B_z (TE) or E_z (TM)
  but check its range of validity for dielectric cavity.
- $\Rightarrow$ Apply Lamé soln., composed from 6 plane waves
  related by 120 rotations. Check TIR confinement...
- The plane waves with smallest angle of incidence can lead to exterior evanescent waves and a lifetime.

triangle with DBC

$$\nabla^2\psi - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0,$$

add 3 'standing waves' related by 120° rotations R

one ray produces 6 travelling waves



$$\psi = \mathcal{A}_0\psi_0 + \mathcal{A}_1\psi_1 + \mathcal{A}_2\psi_2.$$

TE, $\Psi = B_Z$, TM, $\Psi = E_Z$

$$\psi_0 = e^{i\vec{k}_1\cdot\vec{r}}\sin\left[\vec{k}_2\cdot\vec{r} + \frac{k_2 a}{2\sqrt{3}}\right]$$

$$\psi_1 = e^{i(R\vec{k}_1)\cdot\vec{r}}\sin\left[(R\vec{k}_2)\cdot\vec{r} + \frac{k_2 a}{2\sqrt{3}}\right]$$

$$\psi_2 = e^{i(R^2\vec{k}_1)\cdot\vec{r}}\sin\left[(R^2\vec{k}_2)\cdot\vec{r} + \frac{k_2 a}{2\sqrt{3}}\right]$$

Imposing DBC on all boundaries determines the allowed wavevector components as

$$k_1 = \frac{2\pi}{3a} m, \qquad m = 0, 1, 2... \tag{26}$$

$$k_2 = \frac{2\pi}{3a} \sqrt{3}\, n, \qquad n = 1, 2, 3... \tag{27}$$

Furthermore, the parity constraint $e^{i\pi m} = e^{i\pi n}$ appears; that is, n and m are either both odd or both even.

The resulting xy frequencies are given by

$$\omega = c^* \sqrt{k_1^2 + k_2^2} = \frac{c}{\sqrt{\epsilon\mu}} \frac{2\pi}{3a} \sqrt{m^2 + 3n^2} \tag{28}$$

The mode wavefunctions are described completely using the amplitude relationships that result:

$$\mathcal{A}_1 = \mathcal{A}_0 e^{i\frac{2\pi}{3}m}, \qquad \mathcal{A}_2 = \mathcal{A}_0 e^{-i\frac{2\pi}{3}m}. \tag{29}$$

Constraints:

1. m<n
2. m, n both odd or both even

Conditions for confinement by total internal reflection

wave component at smallest angle of incidence has

$\sin\theta_i = k_1/(k_1^2 + k_2^2)^{1/2}$

wavevector is

$(k_1, k_2) = (2\pi/3a)(m, n\sqrt{3})$

TIR confined when

$\sin\theta_i > n'/n = 1/N$

$$\mathrm{N} > \mathrm{N}_c = \sqrt{3\frac{n^2}{m^2} + 1}.$$

or,

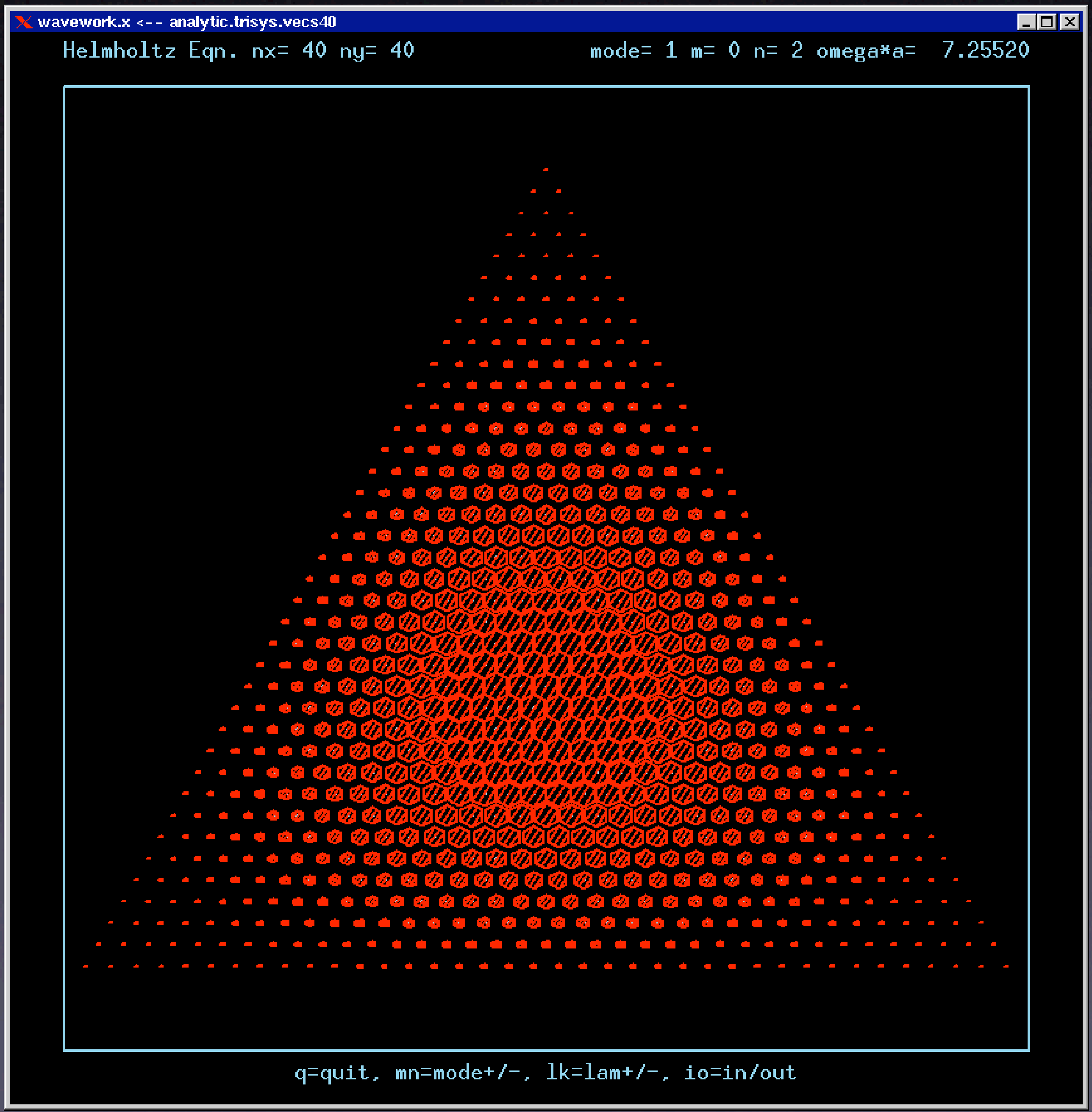
$$\frac{m}{n} > \sqrt{\frac{3}{\mathrm{N}^2 - 1}}.$$

$(m,n)=(0,2)$

$(k_1,k_2)=(2\pi/3a)(0,2\sqrt{3})$

ground state

min. $\sin\theta_i = 0$



Modes with m=0 can never be confined by TIR because $k_1=0$ (m=0)

gives waves at zero incident angle on all boundaries

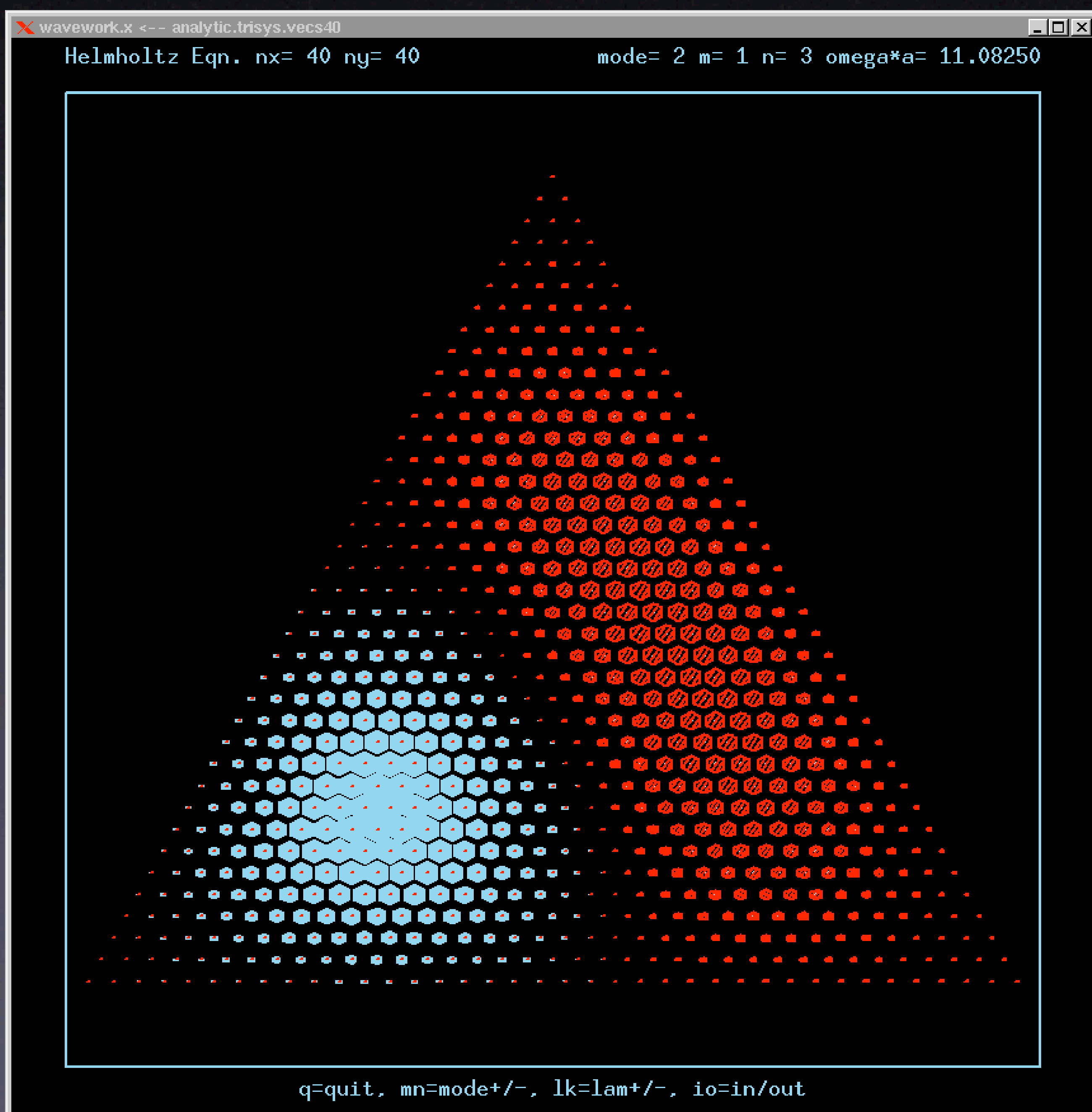
$(m,n)=(1,3)$

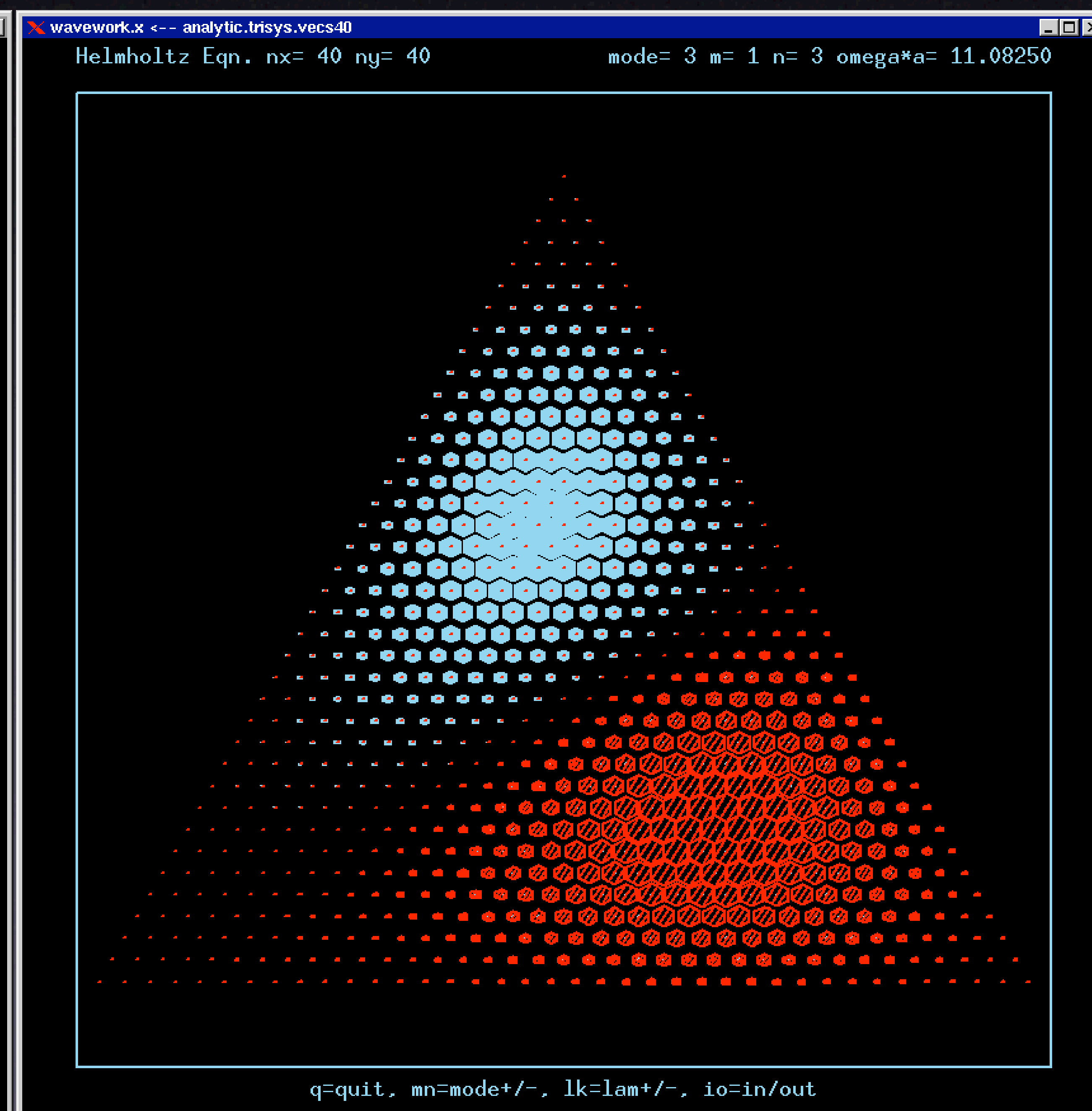
min. $\sin\theta_i = 1/(1^2 + 3\times 3^2)^{1/2}$

$k=(2\pi/3a)(1,3\sqrt{3})$

doubly degenerate

TIR requires N>5.29



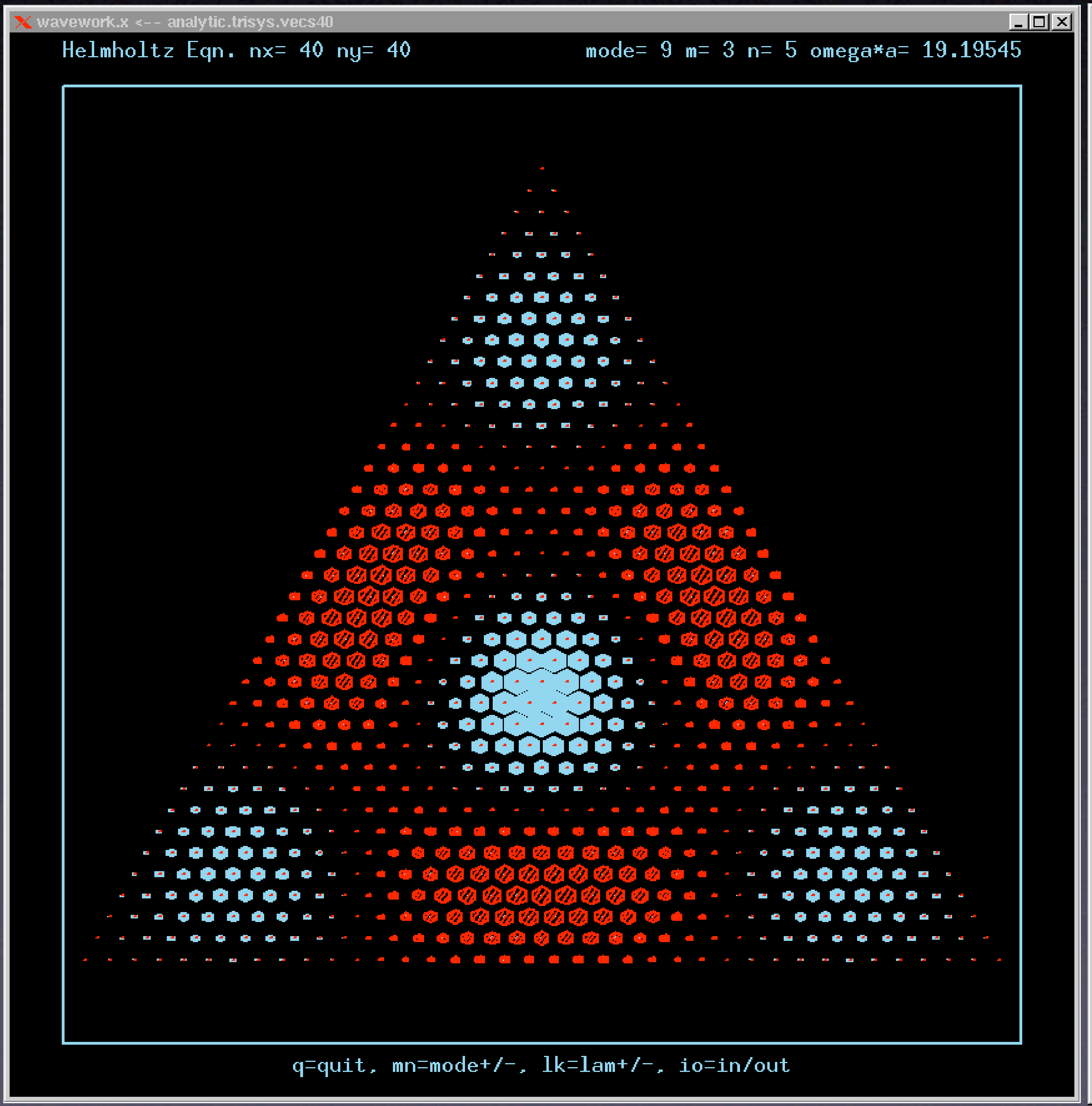


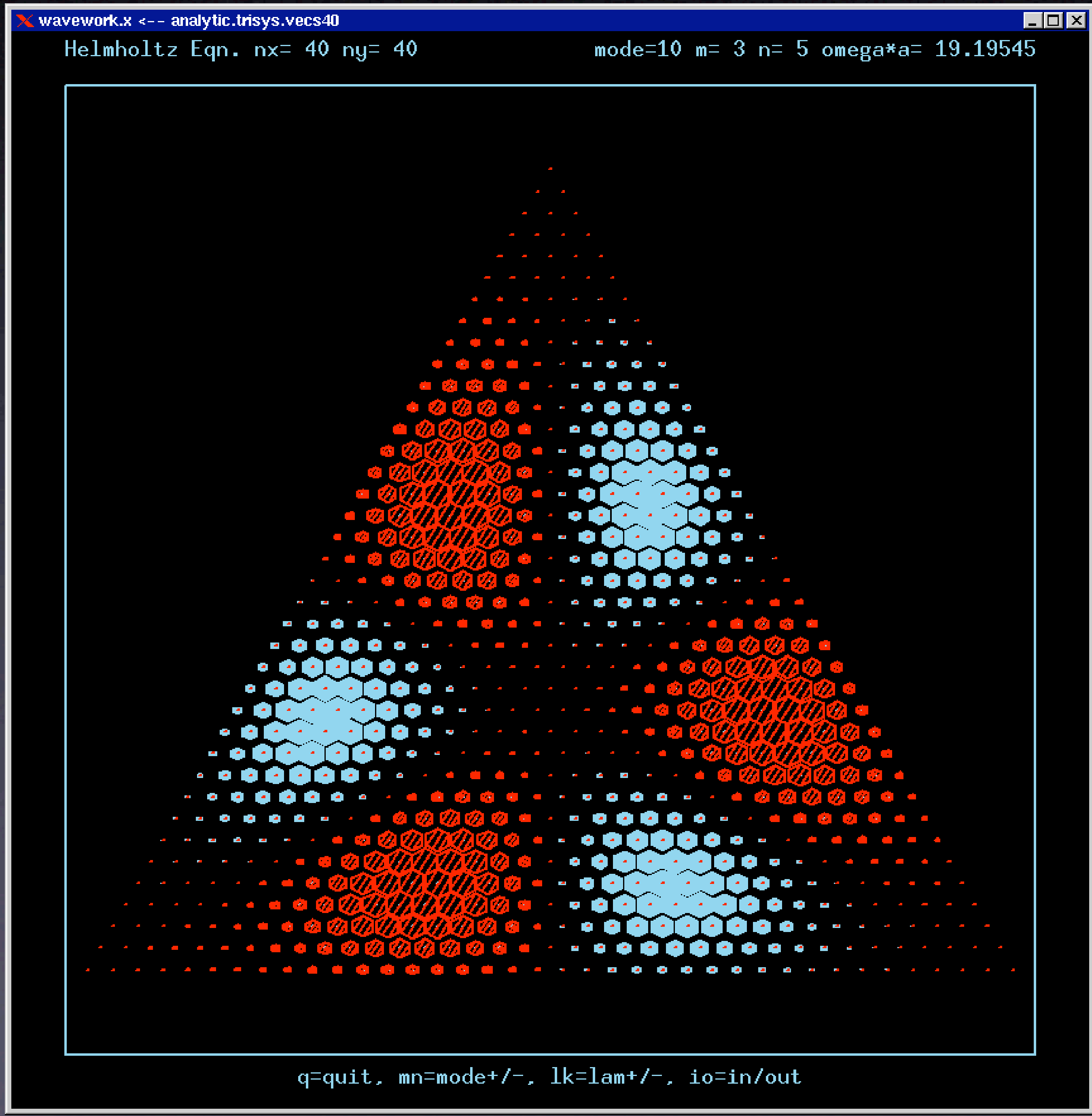
$(m,n)=(3,5)$ min. $\sin\theta_i=3/(3^2+3\times5^2)^{1/2}$

$k=(2\pi/3a)(3,5\sqrt{3})$

doubly degenerate

TIR requires N>3.06





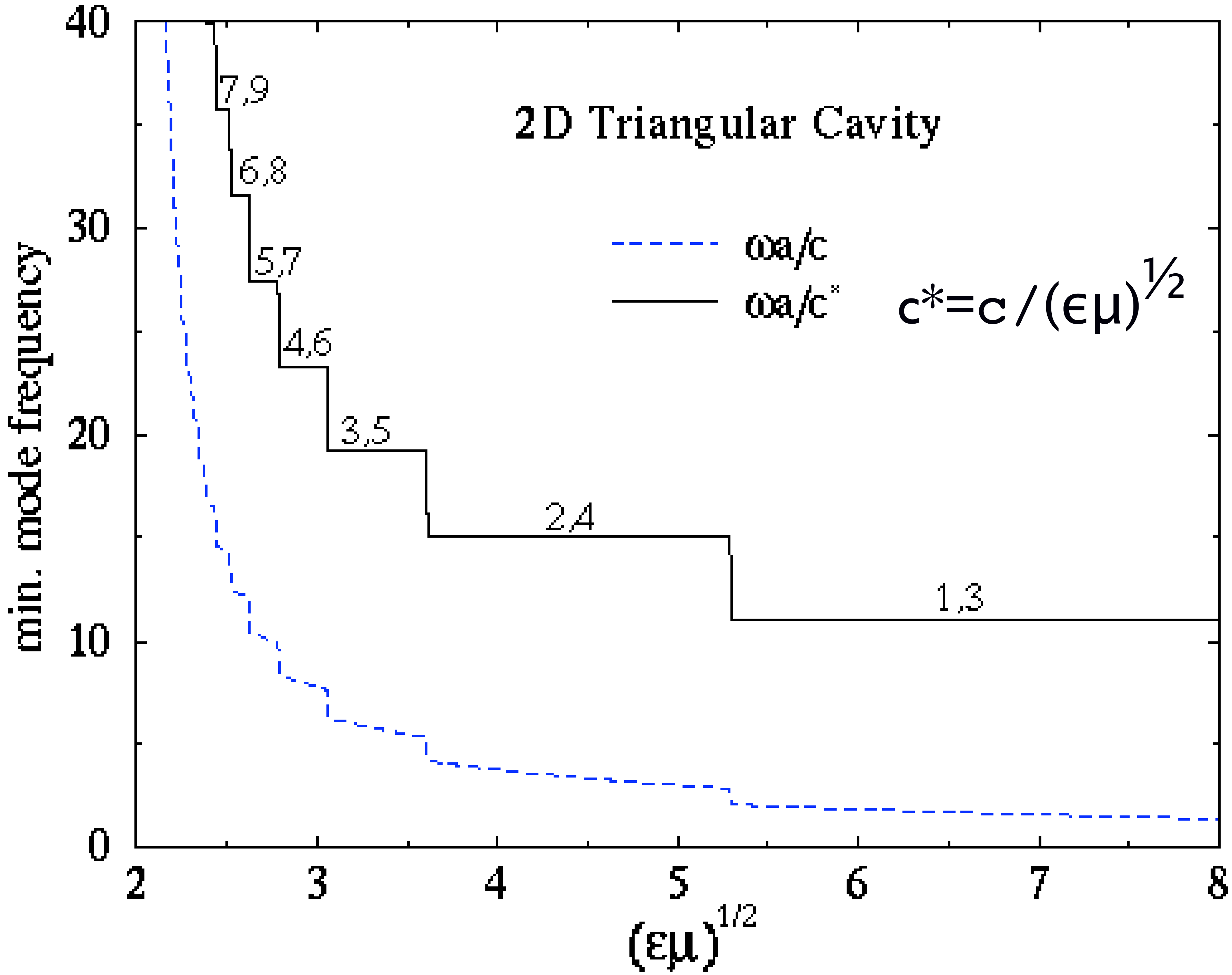
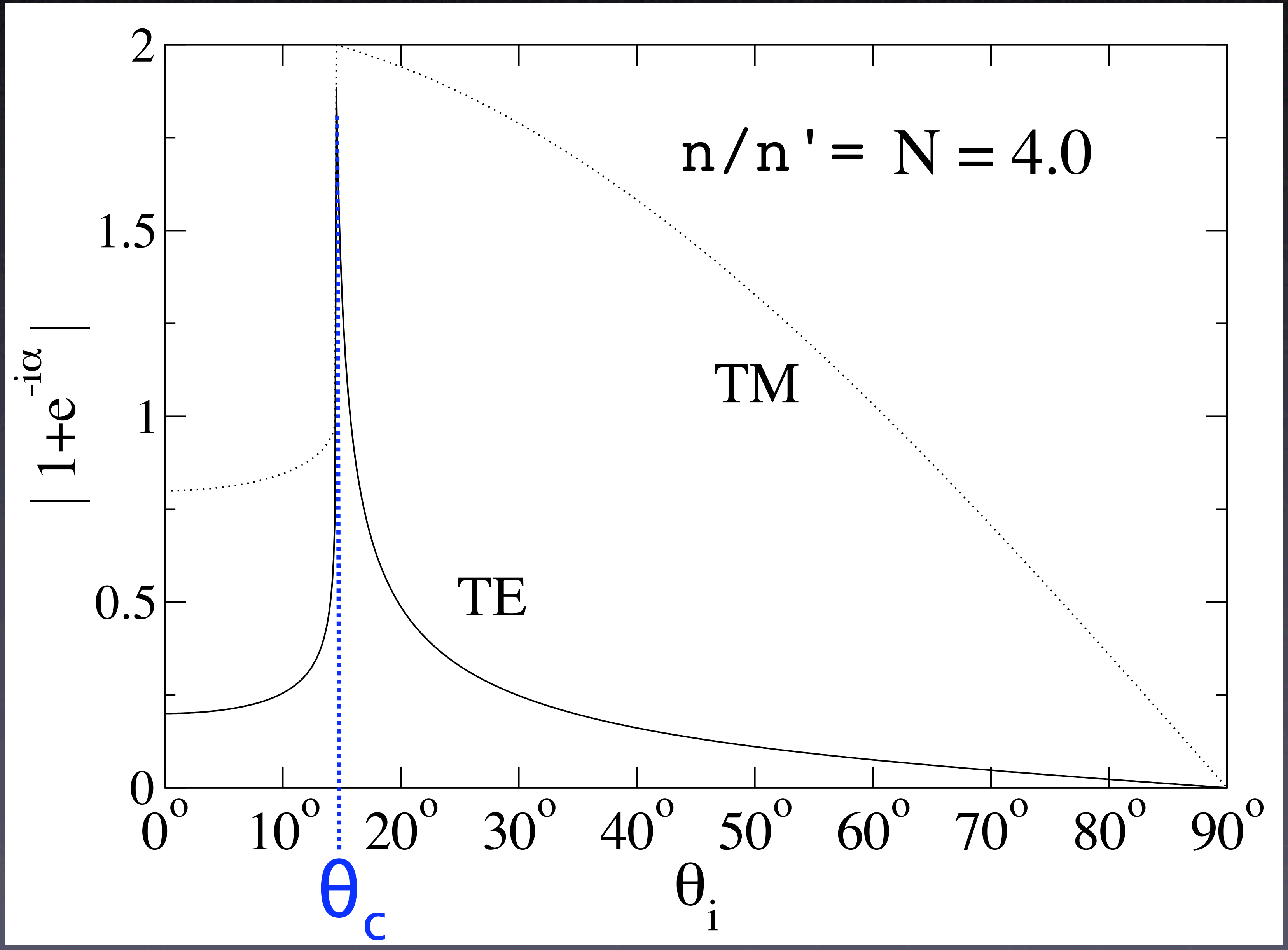


FIG. 7: Frequency of the lowest confined mode for a 2D triangular system surrounded by vacuum, as a function of the refractive index. Pairs (m, n) indicate some of the modes' quantum numbers. No modes are confined for $\sqrt{\epsilon\mu} \leq 2$.

How good is Dirichlet BC approximation?

exterior field at boundary: $\Psi_{ext} \propto (1 + e^{-i\alpha})\Psi_{in}$

$e^{-i\alpha}$ is reflection amplitude of Fresnel equations.



⇐ Example.

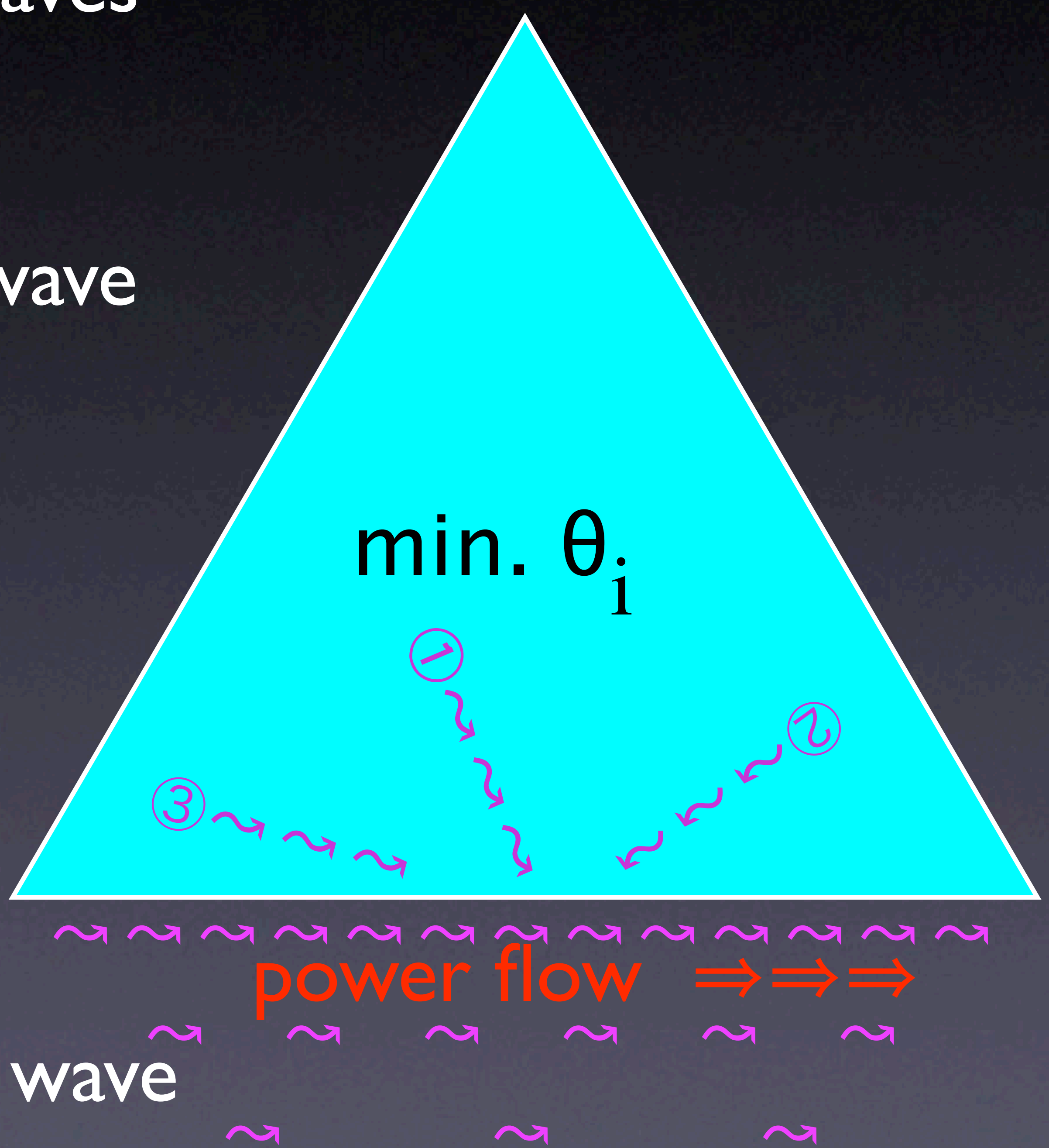
DBC better for modes with large m/n (or θ_i) and much better for TE than TM polarization.

Mode lifetime estimates--due to escape of evanescent boundary waves at the triangle vertices (Wiersig 2003).

DBC solution has three plane waves incident on any boundary.

Wave ① produces evanescent wave of greatest power (min. θ_i)

lifetime $\tau \approx U_{cavity} \div 3P_{①}$



U_{cavity} = total cavity energy

$P_{①}$ = power in ①'s evanescent wave

Results from boundary wave power.

wave ①, $\theta_i = \theta_0^-$

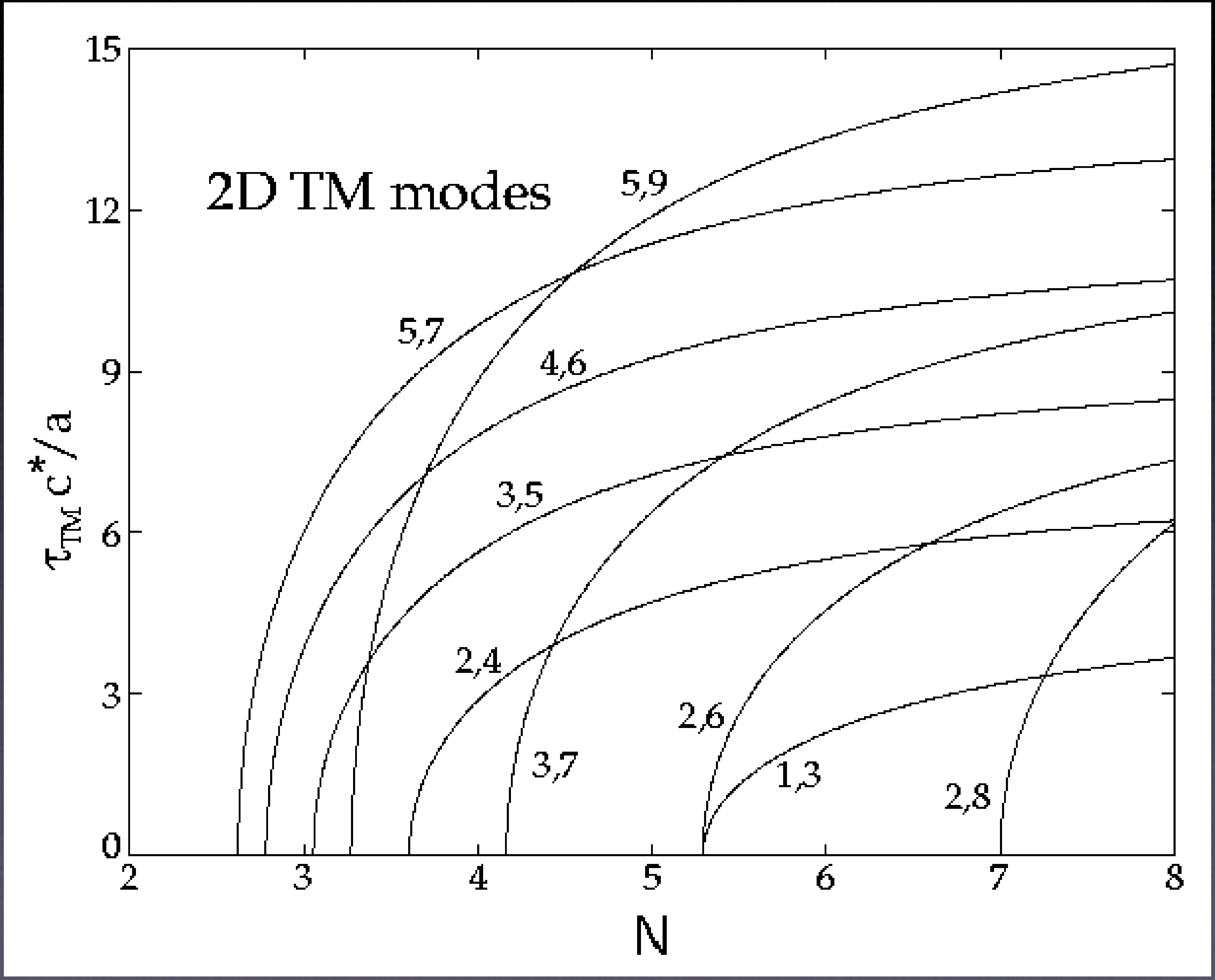
$$Q_{\rm TE} = \omega\tau_{\rm TE} \approx \frac{\sqrt{3}}{4}\left(\frac{\omega a}{c^*}\right)^2 \frac{\sqrt{1-(\sin\theta_c/\sin\theta_0^-)^2}}{\cos^2\theta_0^-} \times \frac{\epsilon}{\epsilon'}\left[\sin^2\theta_0^- - \sin^2\theta_c + \left(\frac{\epsilon'}{\epsilon}\right)^2\cos^2\theta_0^-\right].$$

$$Q_{\rm TM} = \omega\tau_{\rm TM} \approx \frac{\sqrt{3}}{4}\left(\frac{\omega a}{c^*}\right)^2 \frac{\sqrt{1-(\sin\theta_c/\sin\theta_0^-)^2}}{\cos^2\theta_0^-} \times \frac{\mu}{\mu'}\left[\sin^2\theta_0^- - \sin^2\theta_c + \left(\frac{\mu'}{\mu}\right)^2\cos^2\theta_0^-\right],$$

$\cos^2\theta_c$ when $\mu=\mu'$

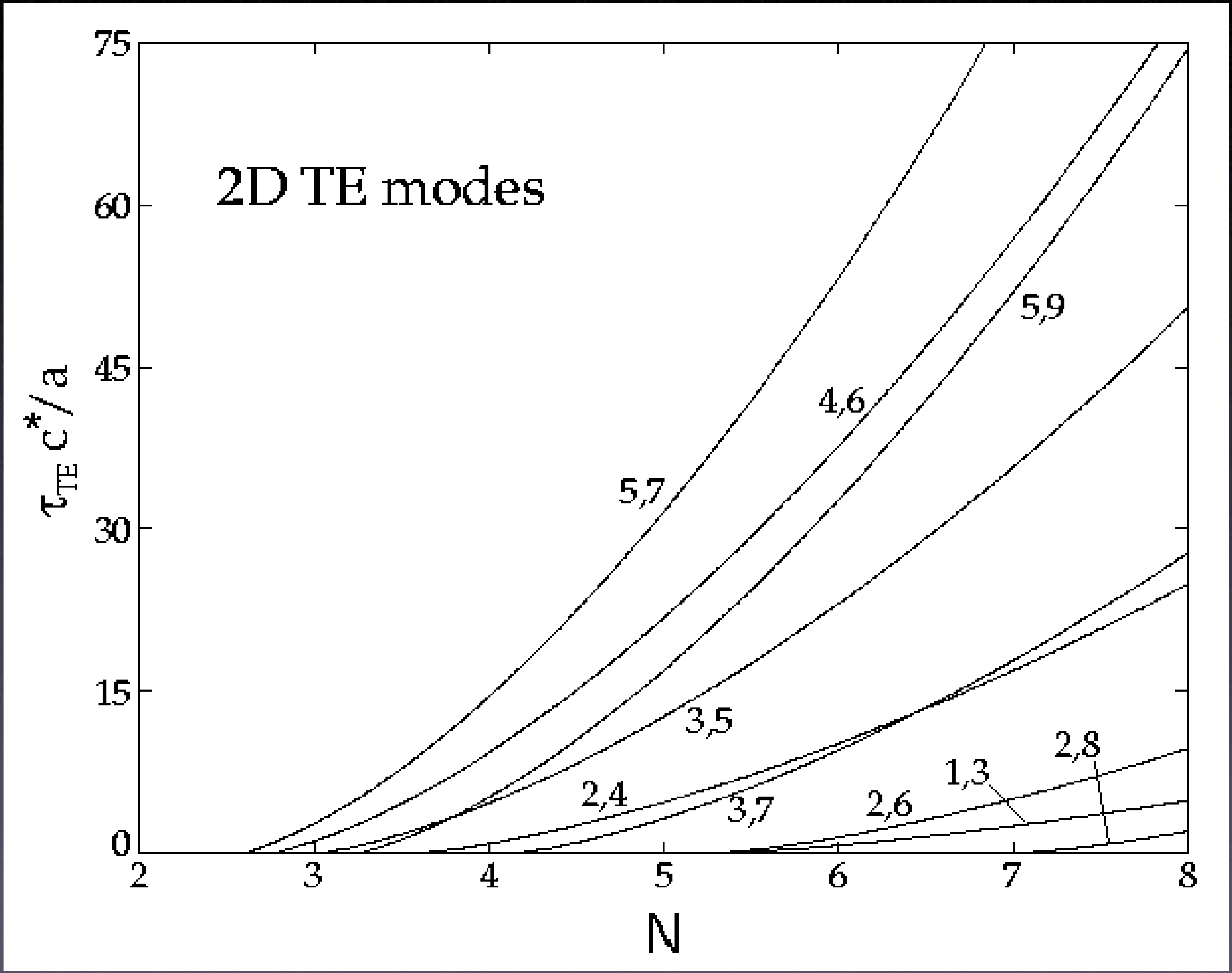
some TM mode lifetimes
vs. index ratio N=n/n'

$\tau_{TM} \sim (\varepsilon\mu)^{1/2}(a/c)$

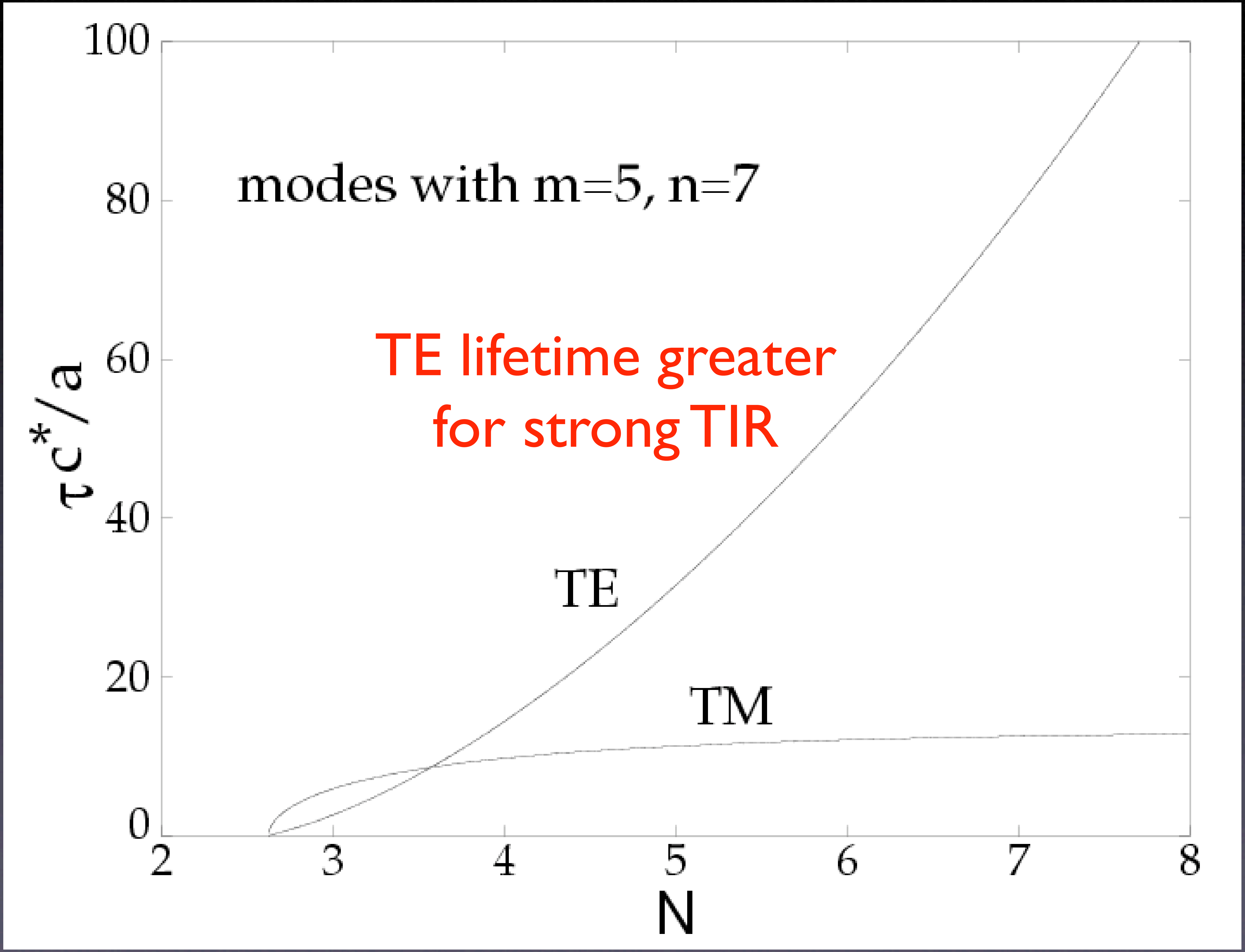


some TE mode lifetimes vs. index ratio N=n/n'

$\tau_{TE} \sim (\varepsilon/\varepsilon')(\varepsilon\mu)^{1/2}(a/c)$



mode (5,7) lifetimes vs. index ratio N=n/n'



Conclusions

- Applied analytic soln. of Helmholtz eqn. in an equilateral triangle with $\Psi=0$ on boundary, for both TE and TM polarizations.
- Estimated mode frequencies, symmetries, and index ratios for TIR confinement.
- Assumed the plane wave component with smallest incident angle leaks out of cavity via evanescent boundary waves.
- Estimated TE and TM mode lifetimes.