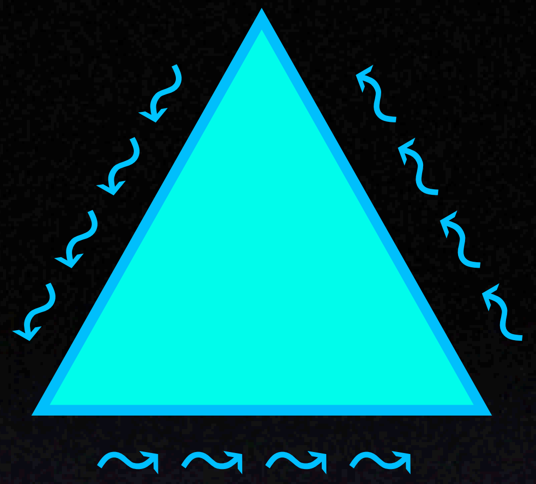
Electromagnetic modes in equilateral triangle dielectric resonators



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Universidade Federal de Viçosa, Minas Gerais, Aug.04-Feb.05
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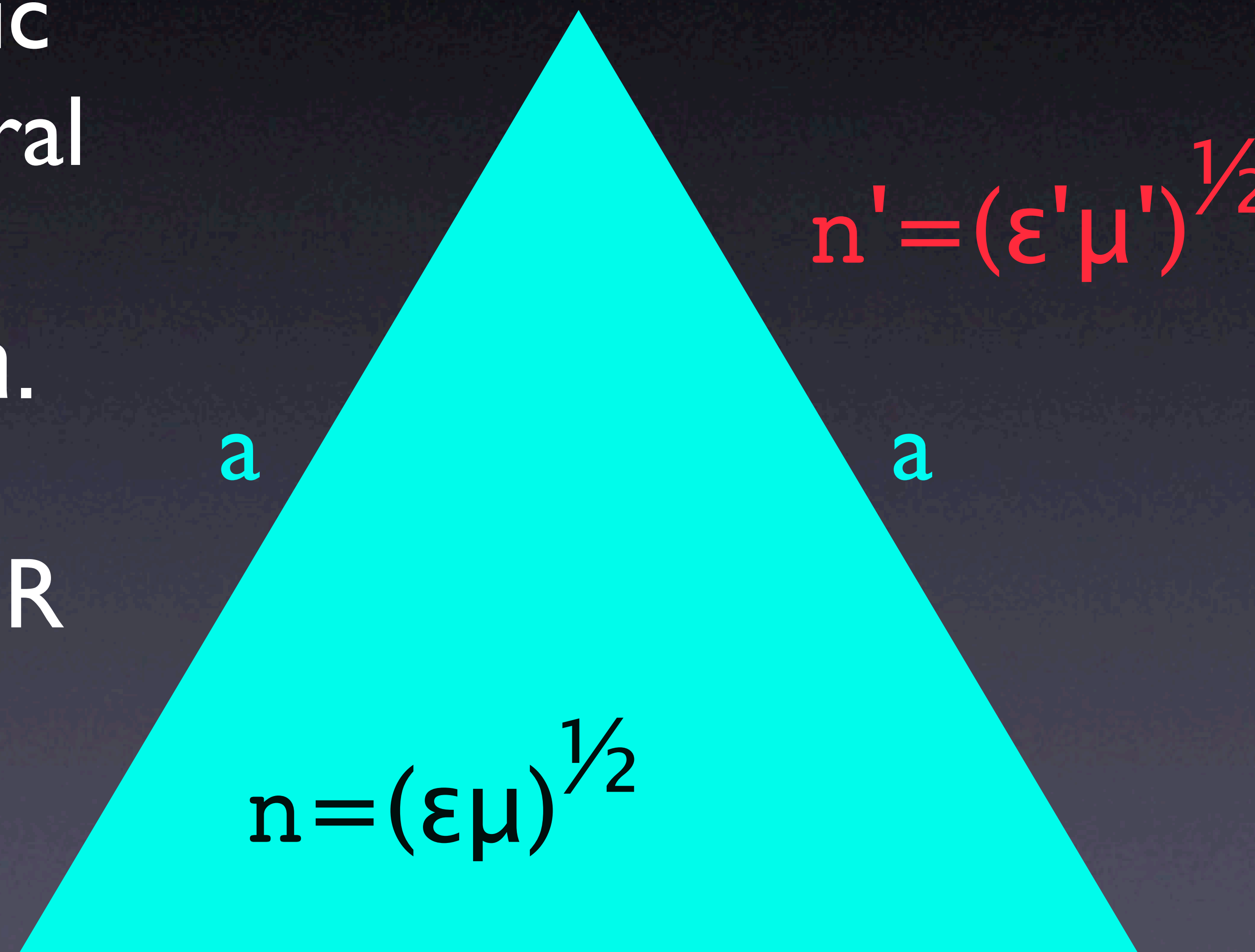
An interesting problem in theoretical and applied physics

- high index dielectric resonator, equilateral triangular cross section, height h≫a.
- what are the 2D TIR resonant modes?
- what are their lifetimes or Q-

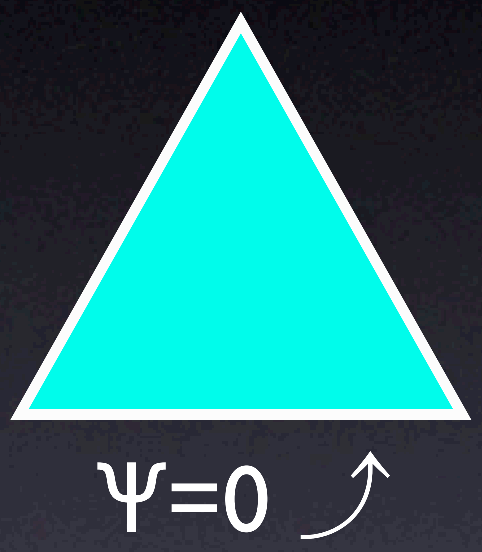


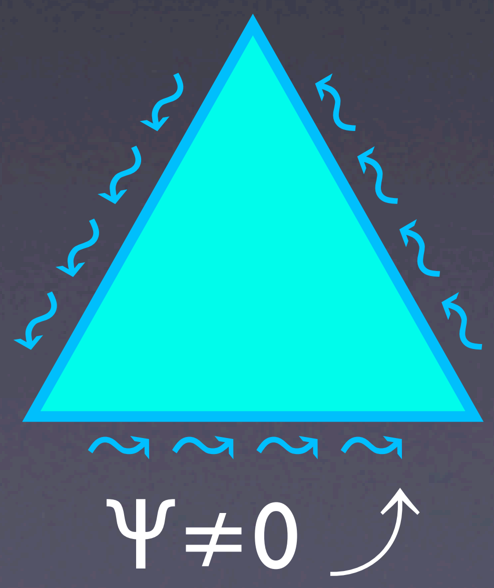
Motivation for studies

- mode selection in micro-cavity resonators
- advances in growth of semiconductor geometries -- circles, triangles, squares, hexagons, pyramids, etc., experimental data
- challenging and fun problem in electromagnetism, field matching conditions, polarization dependence
- relation to quantum polygonal billiards

Some previous works on ETRs

- M. G. Lamé (1852). Analytic soln. for elastic waves on a triangular drum, Helmholtz eq. with $\Psi=0$ on boundary (scalar Ψ, Dirichlet Boundary Conditions)
- H. C. Chang et al. (2000). Used Lamé soln. for TM modes in dielectric ETR, applying DBC.
- Y. Z. Huang et al. (1999-2001). Approx. soln. for TM and TE modes using Maxwell eqs. and their boundary conditions in a dieletric ETR. Evanescent exterior waves.





PART I. Review of Dirichlet Boundary Condition approximation ($\Psi_{edge}=0$)

- Large index ratio $N \equiv n/n' \gg 1$, $\Rightarrow$ strong TIR.
- No z-dependence; independent TE and TM modes.
- $\boxed{\text{Initial approx: DBC on } \Psi=H_z \text{ (TE) or } \Psi=E_z \text{ (TM)}}$
- $\Rightarrow$ Apply Lamé soln., composed from 6 plane waves related by 120 rotations. Check TIR confinement...
- The plane waves with smallest angle of incidence can lead to exterior evanescent waves and a lifetime.

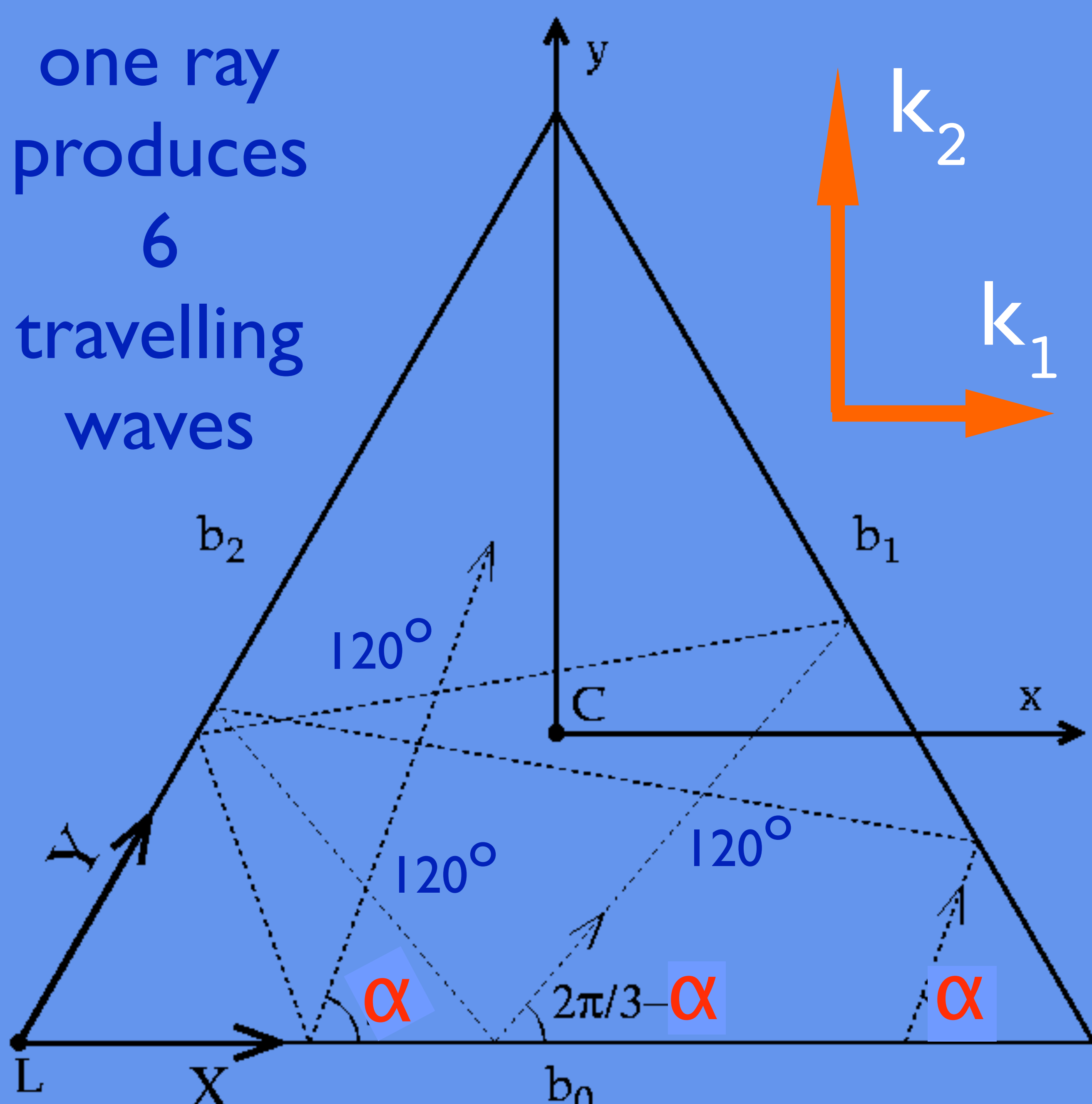
triangle with DBC

$$\nabla^2\psi - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0,$$

add 3 'standing waves'

related by 120° rotations R

one ray produces 6 travelling waves



$$\psi = \mathcal{A}_0\psi_0 + \mathcal{A}_1\psi_1 + \mathcal{A}_2\psi_2.$$

TE, $\Psi = H_z$, TM, $\Psi = E_z$

$$\psi_0 = e^{i\vec{k}_1\cdot\vec{r}}\sin\left[\vec{k}_2\cdot\vec{r} + \frac{k_2 a}{2\sqrt{3}}\right]$$

$$\psi_1 = e^{i(R\vec{k}_1)\cdot\vec{r}}\sin\left[(R\vec{k}_2)\cdot\vec{r} + \frac{k_2 a}{2\sqrt{3}}\right]$$

$$\psi_2 = e^{i(R^2\vec{k}_1)\cdot\vec{r}}\sin\left[(R^2\vec{k}_2)\cdot\vec{r} + \frac{k_2 a}{2\sqrt{3}}\right]$$

Imposing DBC on all boundaries determines the allowed wavevector components as

$$k_1 = \frac{2\pi}{3a} m, \qquad m = 0, 1, 2... \tag{26}$$

$$k_2 = \frac{2\pi}{3a} \sqrt{3}\, n, \qquad n = 1, 2, 3... \tag{27}$$

Furthermore, the parity constraint $e^{i\pi m} = e^{i\pi n}$ appears; that is, n and m are either both odd or both even.

The resulting xy frequencies are given by

$$\omega = c^* \sqrt{k_1^2 + k_2^2} = \frac{c}{\sqrt{\epsilon\mu}} \frac{2\pi}{3a} \sqrt{m^2 + 3n^2} \tag{28}$$

The mode wavefunctions are described completely using the amplitude relationships that result:

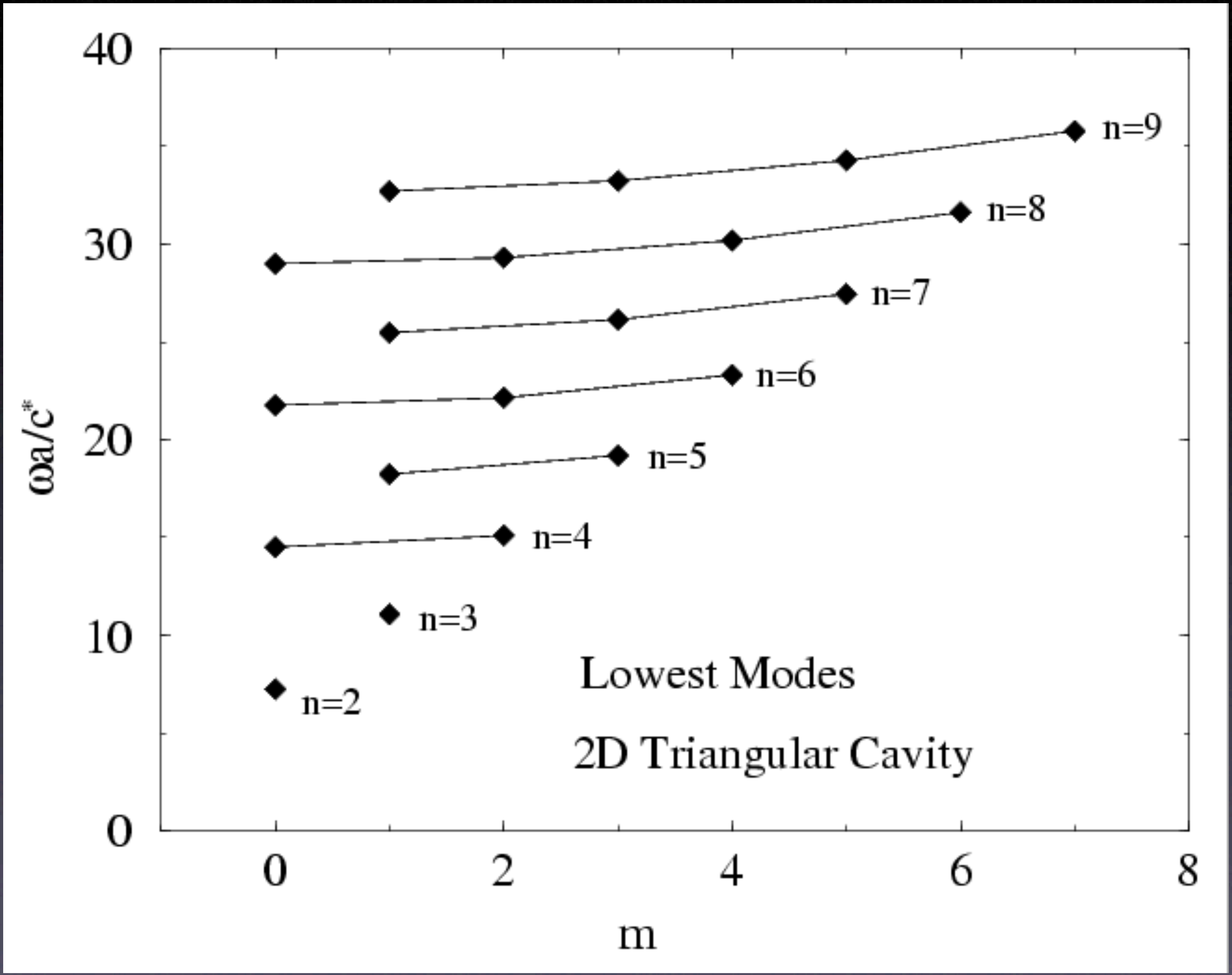
$$\mathcal{A}_1 = \mathcal{A}_0 e^{i\frac{2\pi}{3}m}, \qquad \mathcal{A}_2 = \mathcal{A}_0 e^{-i\frac{2\pi}{3}m}. \tag{29}$$

Constraints:

1. m<n
2. m, n both odd or both even

DBC frequencies

Spectrum with DBC approximation



Conditions for confinement by Total Internal Reflection

wave component at smallest angle of incidence has
$\sin\theta_i = k_1/(k_1^2 + k_2^2)^{1/2}$

wavevector is
$(k_1,k_2) = (2\pi/3a)(m, n\sqrt{3})$

TIR confined when
$\sin\theta_i > n'/n = 1/N$

$$\boxed{N > N_c = \sqrt{3\frac{n^2}{m^2} + 1}.}$$

or,

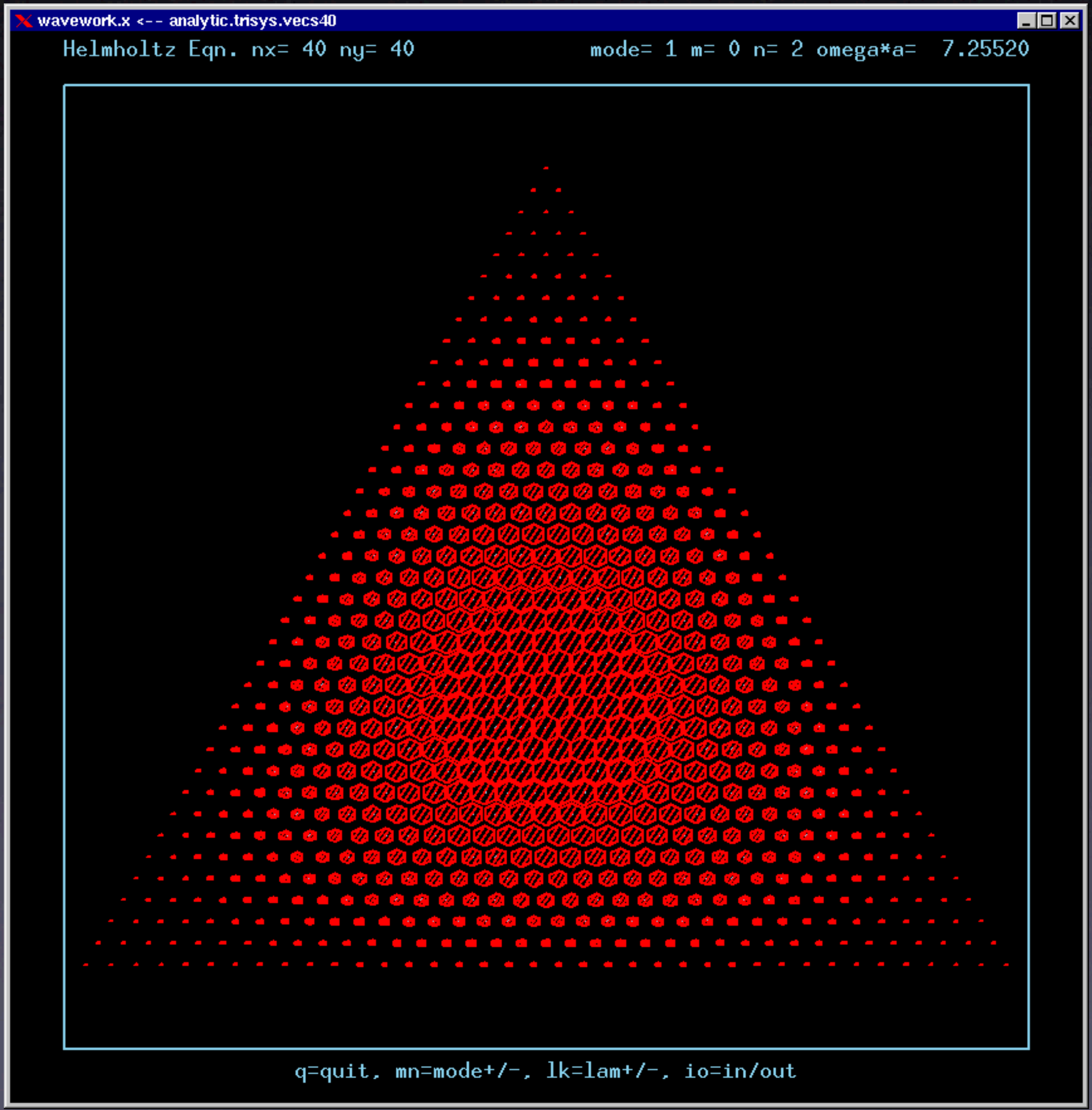
$$\boxed{\frac{m}{n} > \sqrt{\frac{3}{N^2 - 1}}.}$$

$(m,n)=(0,2)$

$(k_1,k_2)=(2\pi/3a)(0,2\sqrt{3})$

ground state

min. $\sin\theta_i = 0$



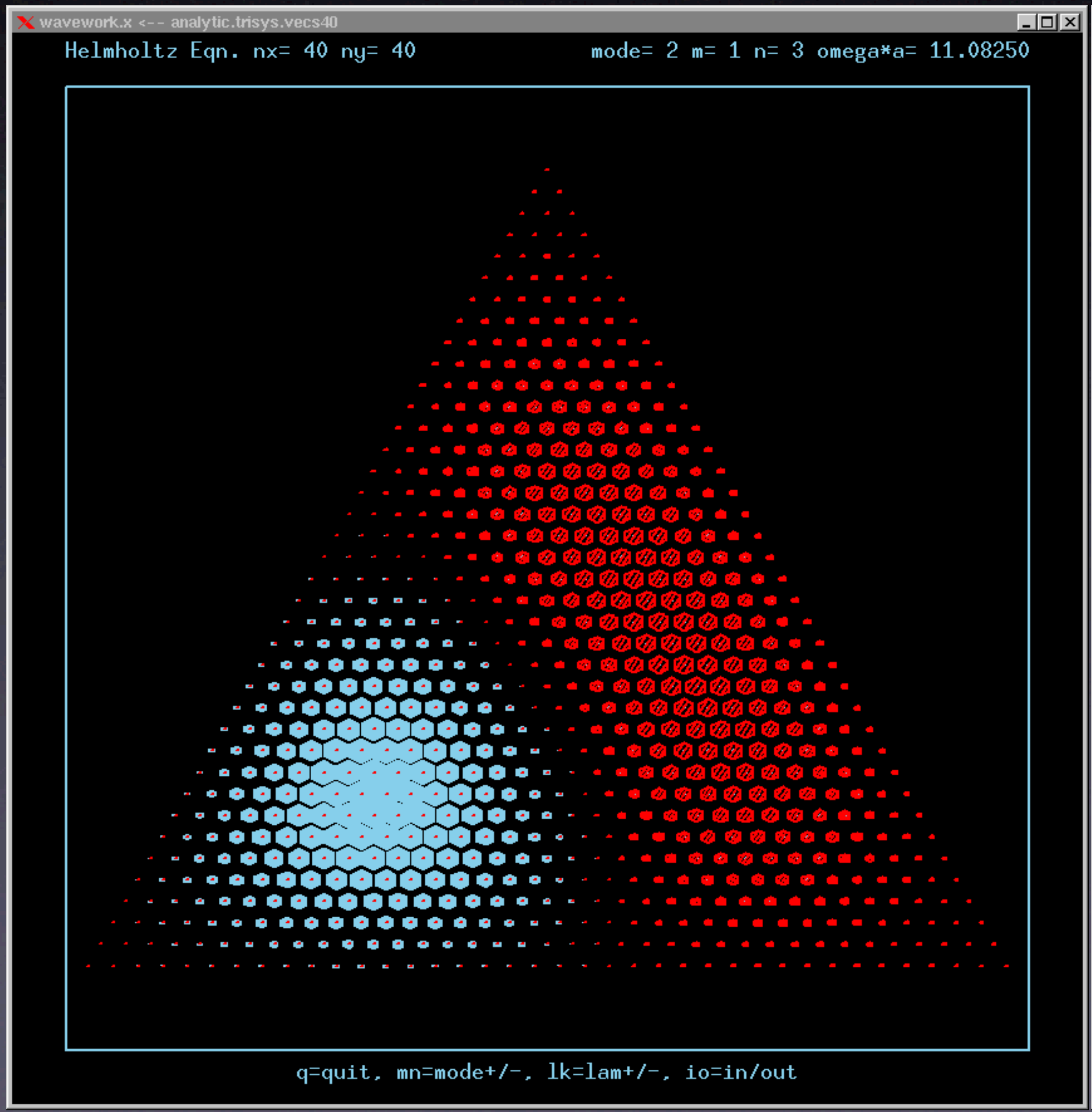
Modes with m=0 can never be confined by TIR because $k_1=0$ (m=0)

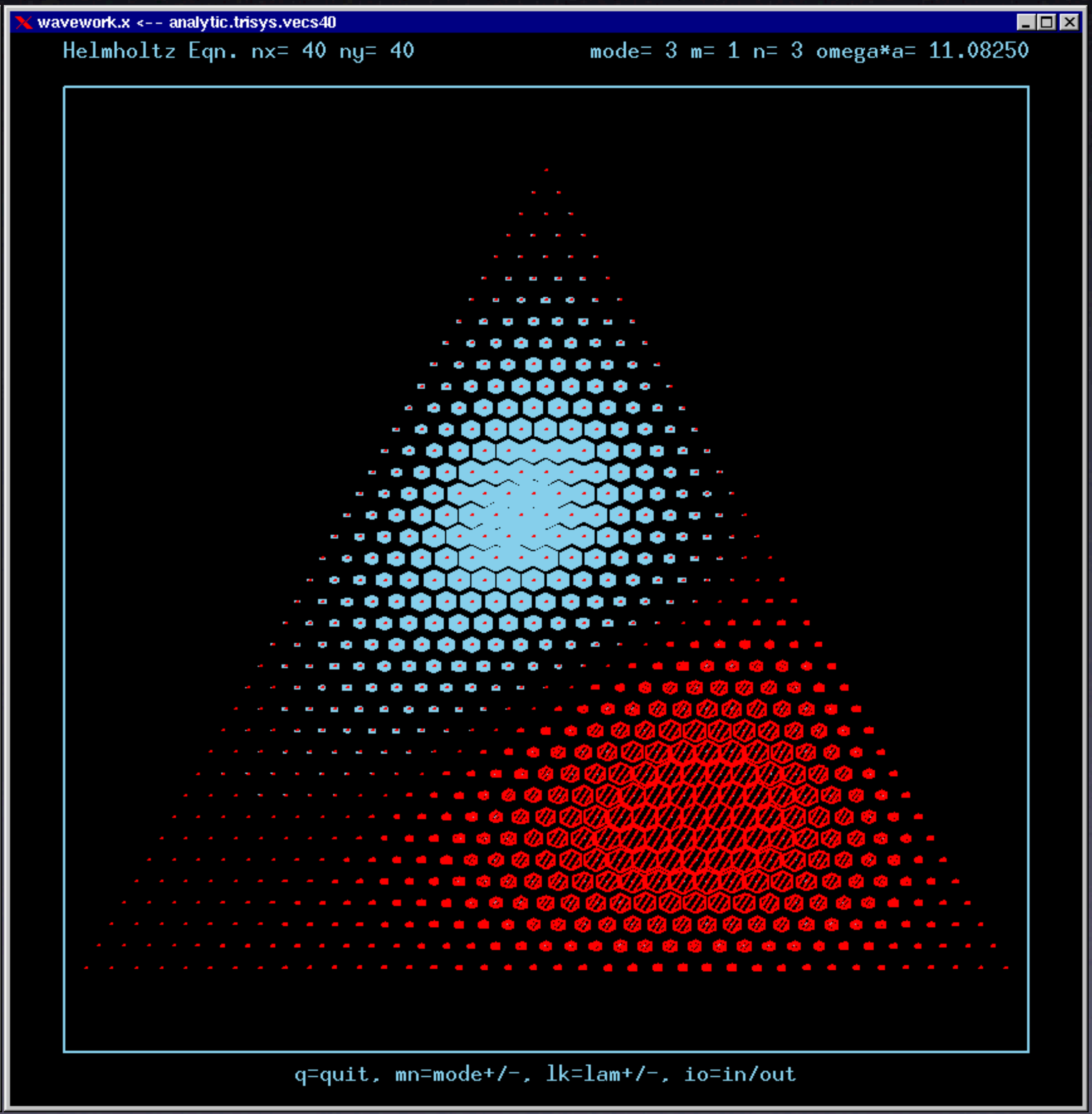
gives waves at zero incident angle on all boundaries

$(m,n)=(1,3)$ min. $\sin\theta_i = 1/(1^2 + 3\times 3^2)^{1/2}$

$k=(2\pi/3a)(1,3\sqrt{3})$
doubly degenerate

TIR requires N>5.29



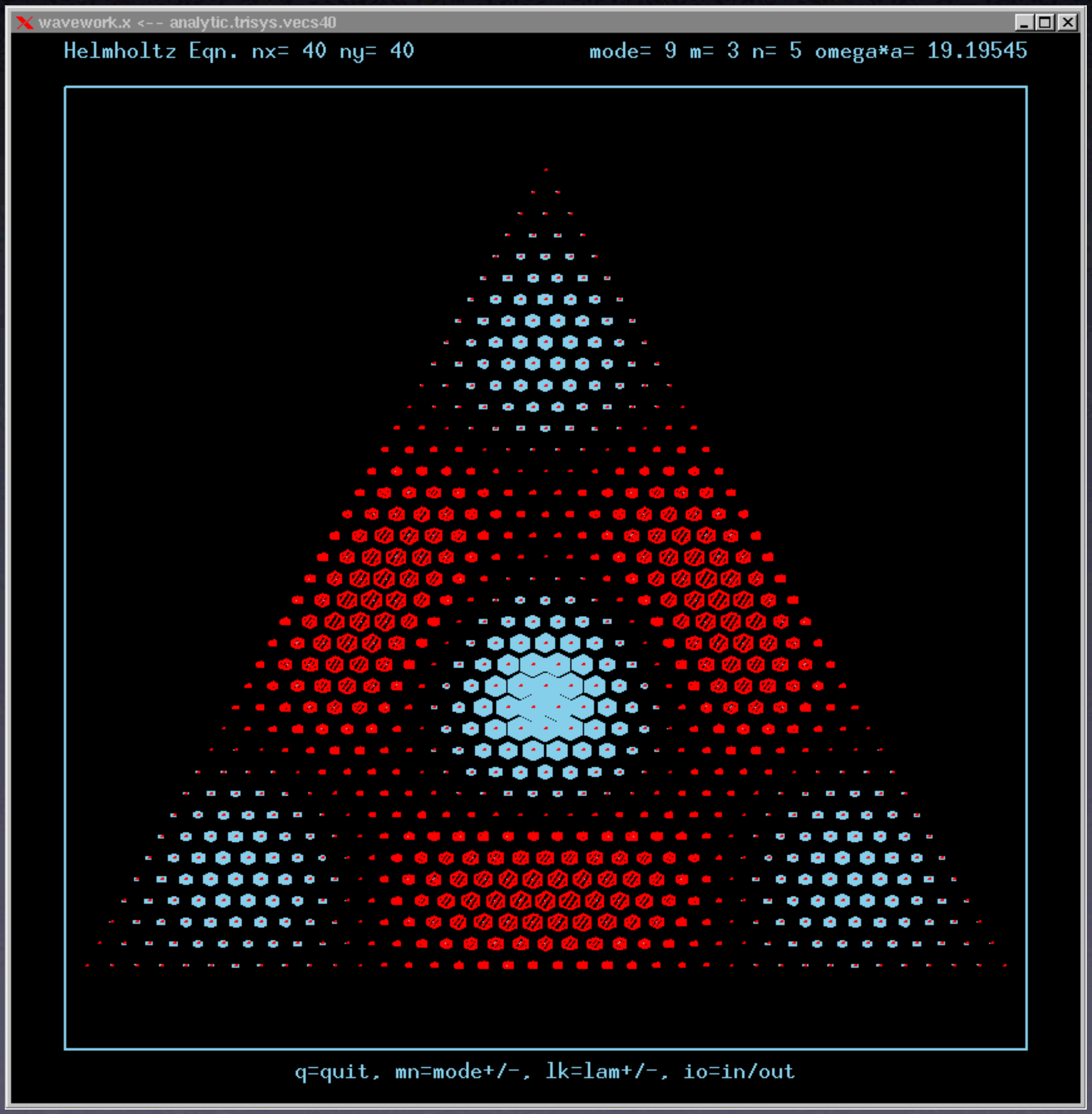


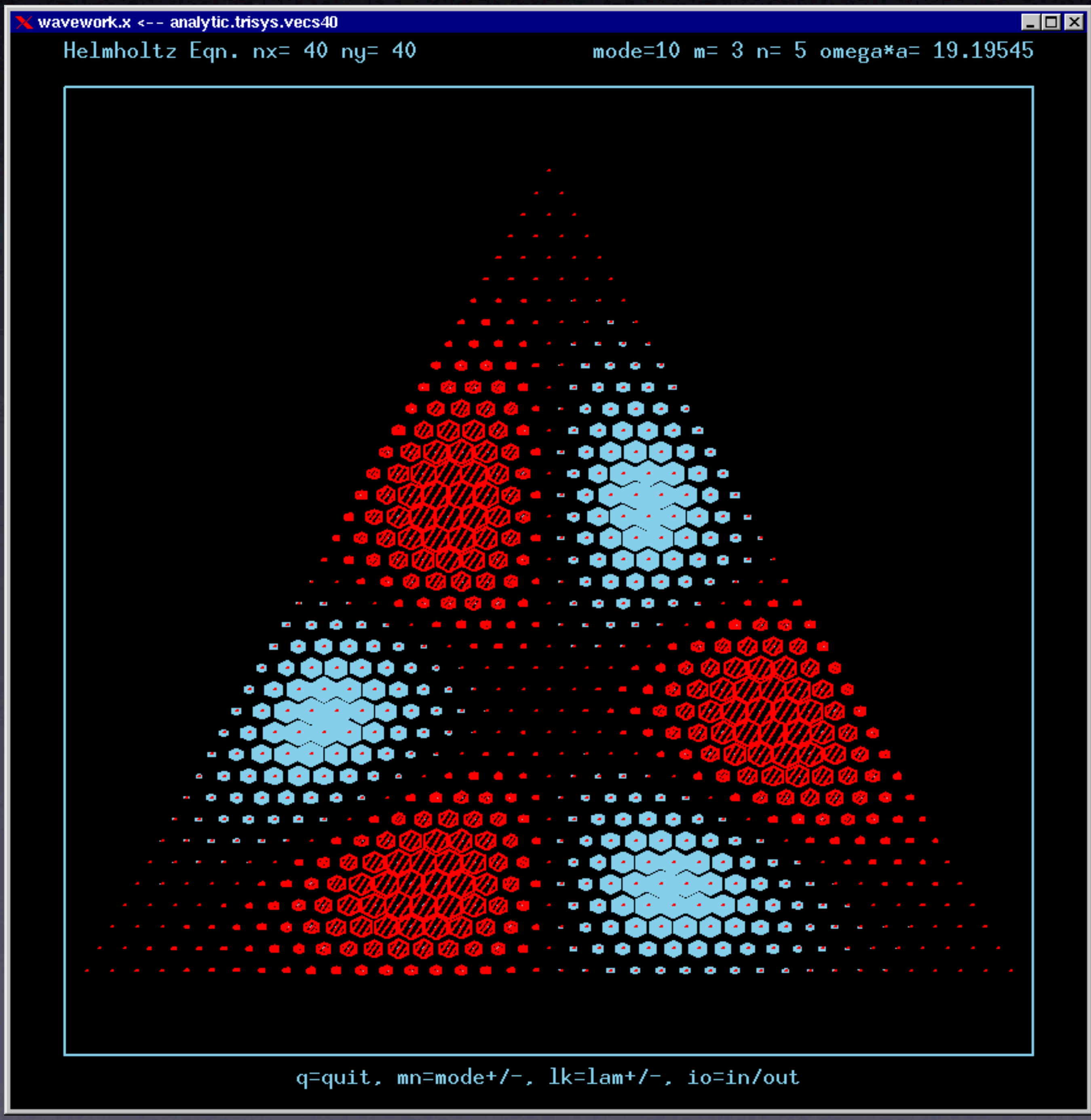
$(m,n)=(3,5)$ min. $\sin\theta_i = 3/(3^2 + 3\times5^2)^{1/2}$

$k=(2\pi/3a)(3,5\sqrt{3})$

doubly degenerate

TIR requires N>3.06





TIR of DBC modes requires:

$$\frac{m}{n} > \sqrt{\frac{3}{\mathsf{N}^2 - 1}}.$$

(and m<n)

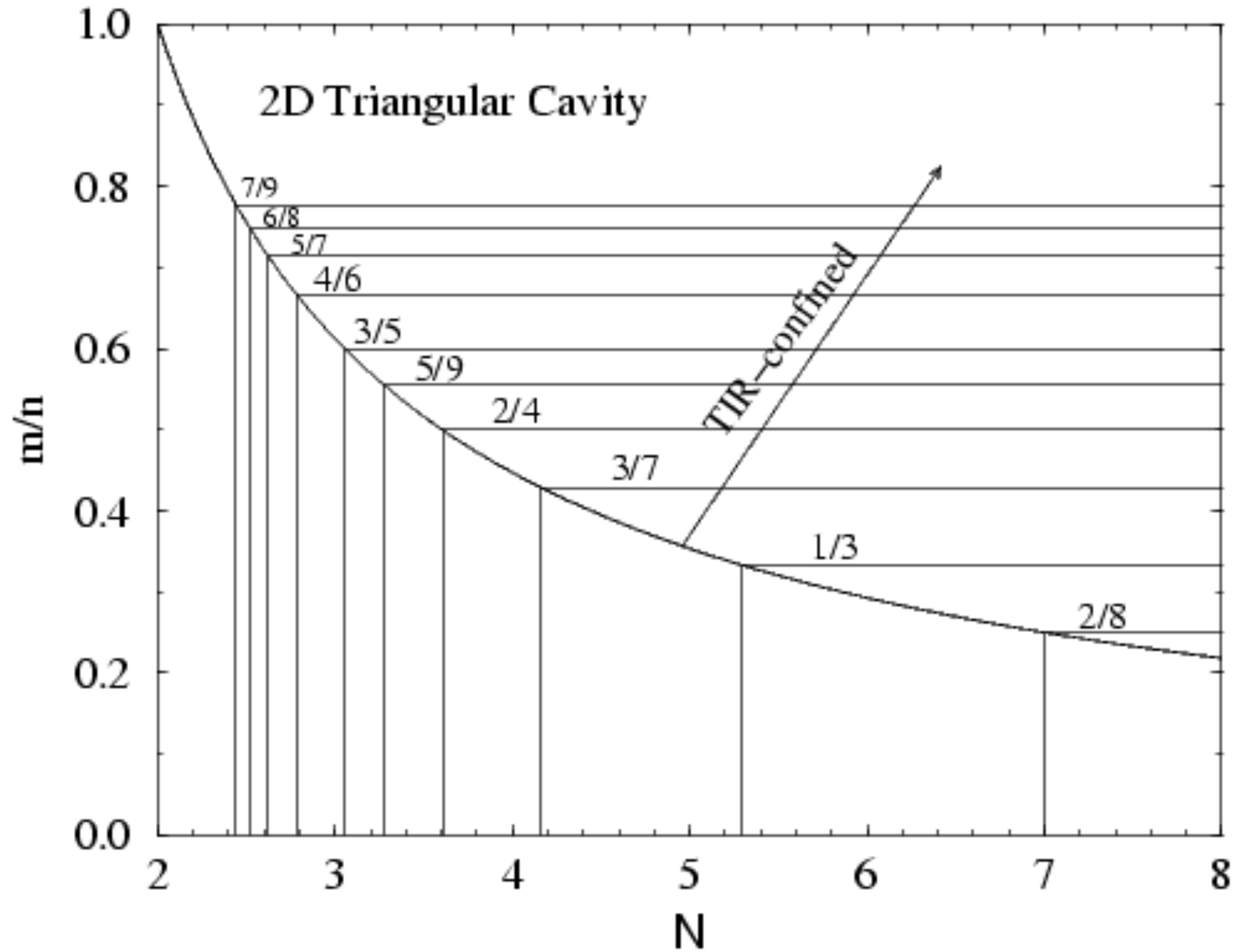


FIG. 6: TIR mode confinement limits (index ratio $\mathsf{N} = \sqrt{\epsilon\mu/\epsilon'\mu'}$) for 2D triangular cavities. Modes are confined where the m/n ratios (as indicated) lie above the solid curve, corresponding to TIR on all boundaries [relation (37)]. Intersections on the N-axis give the critical index ratios for each mode.

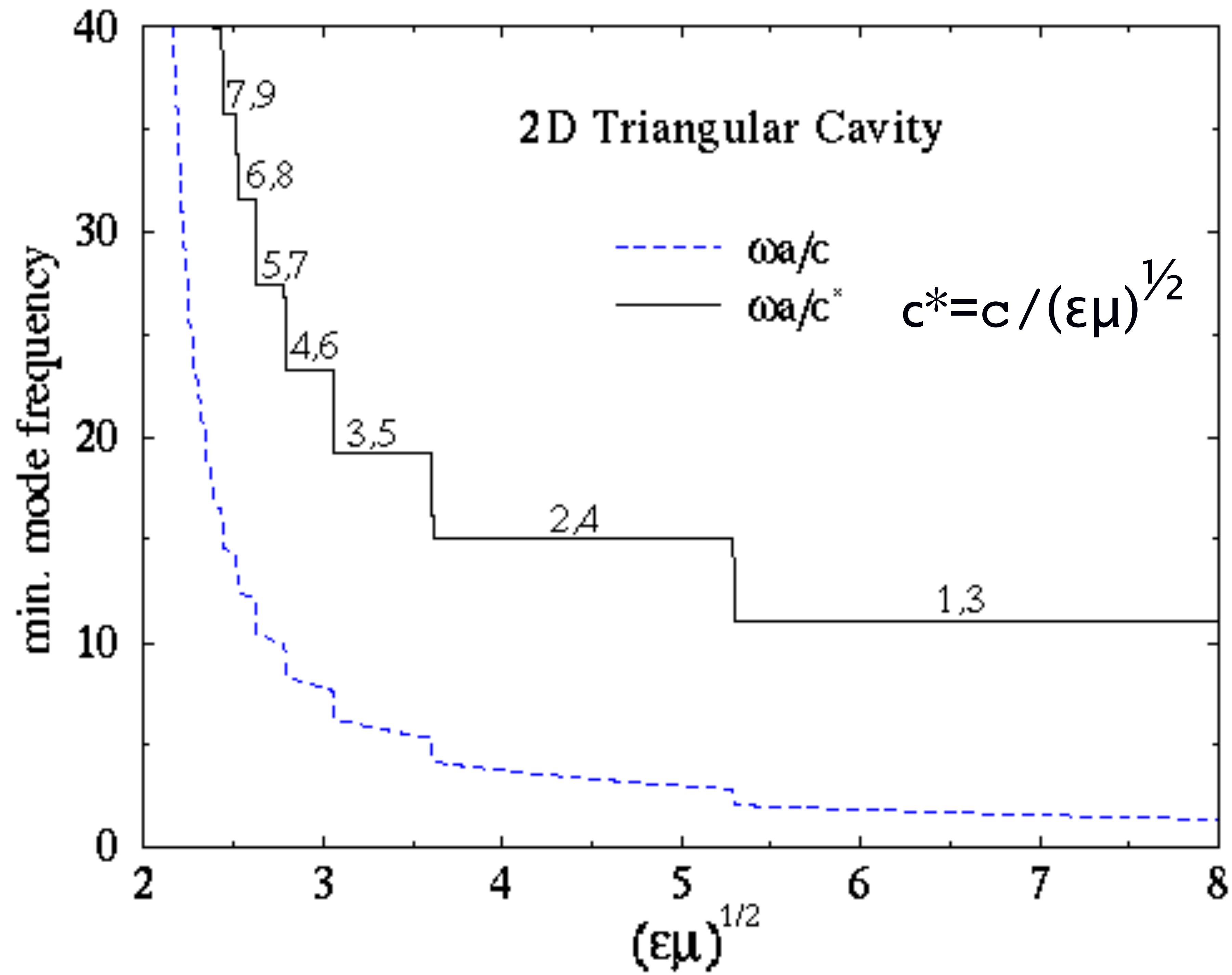
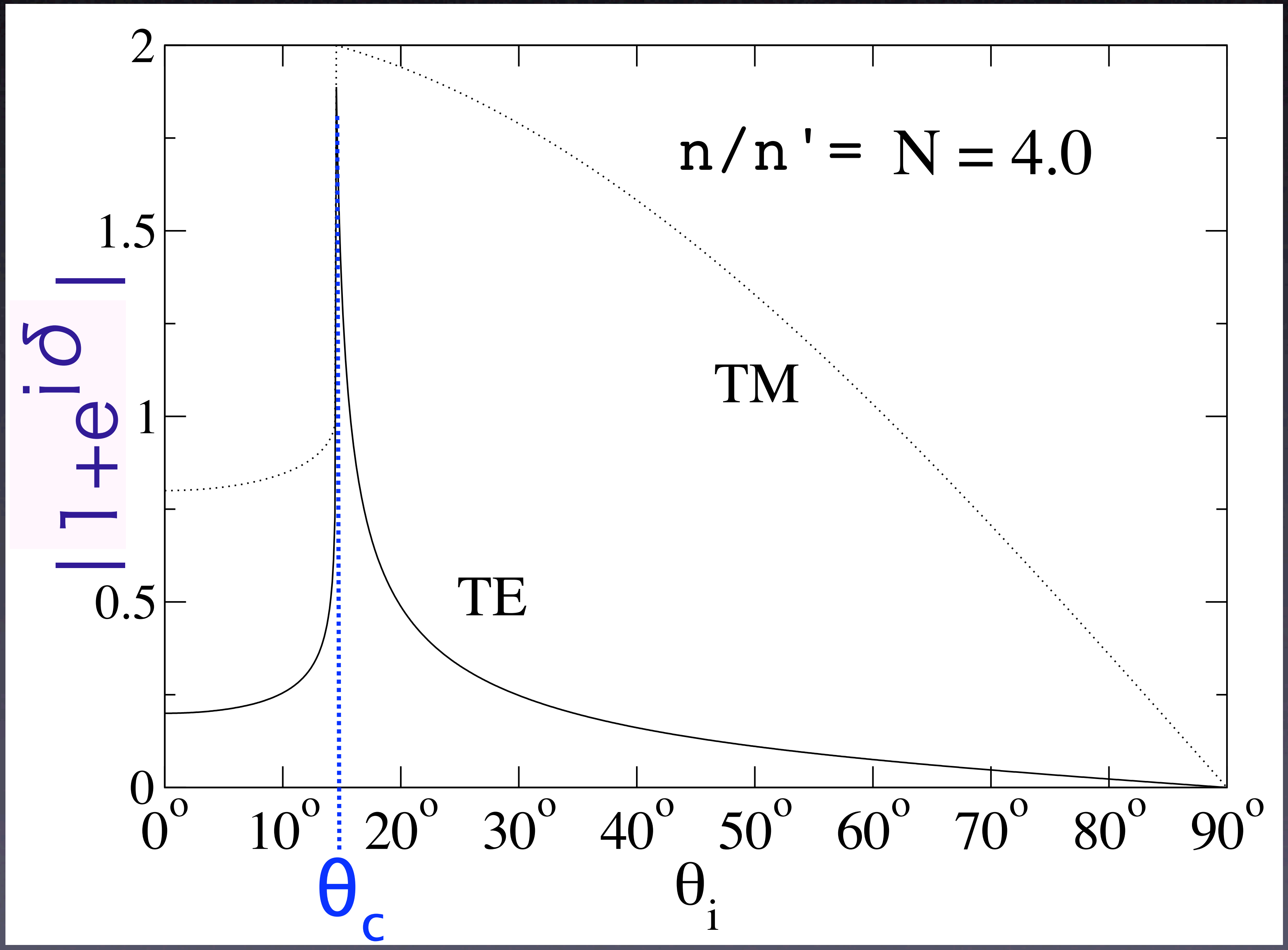


FIG. 7: Frequency of the lowest confined mode for a 2D triangular system surrounded by vacuum, as a function of the refractive index. Pairs (m, n) indicate some of the modes' quantum numbers. No modes are confined for $\sqrt{\epsilon\mu} \leq 2$.

How good is Dirichlet BC approximation?

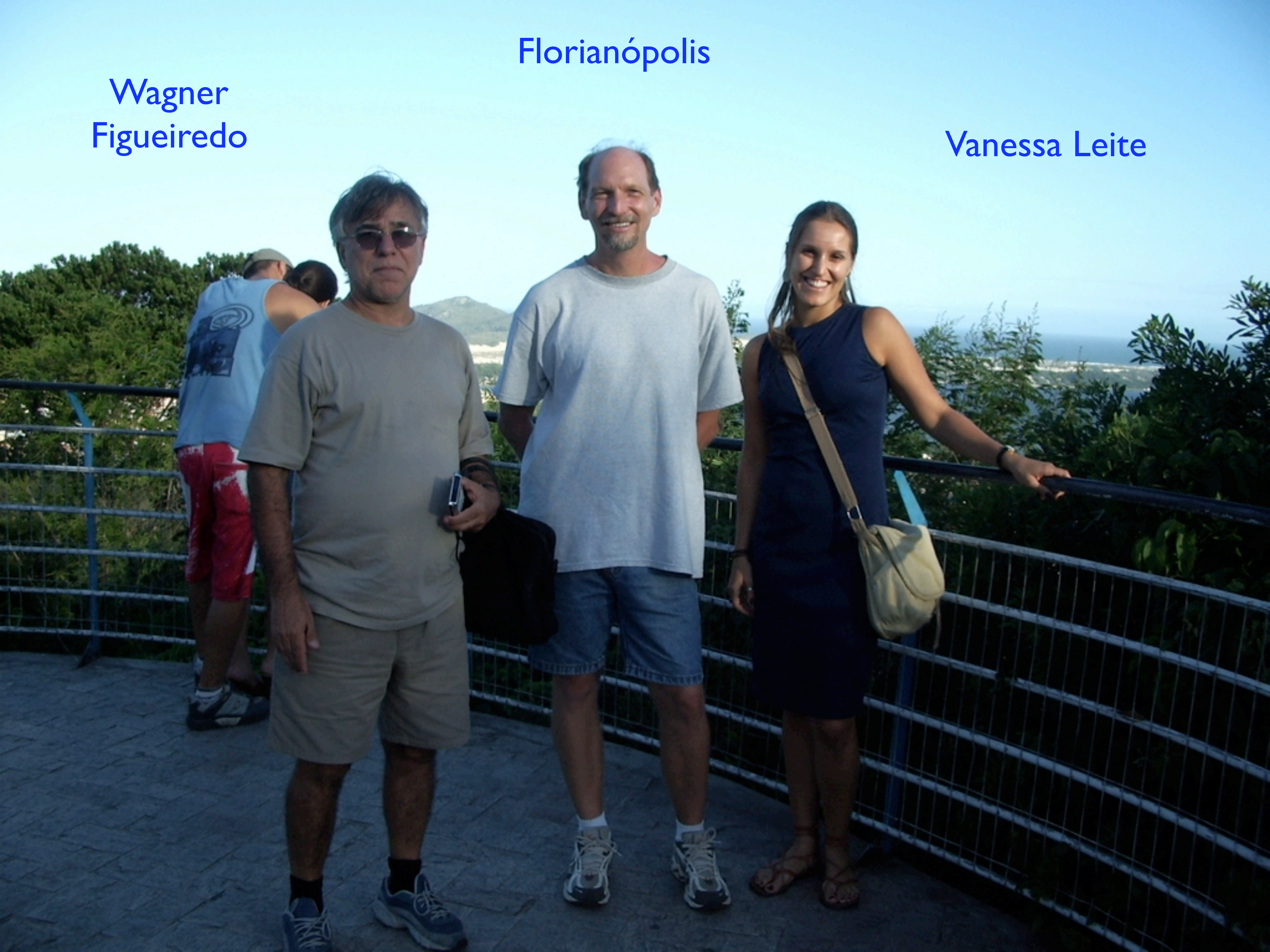
Exterior field at boundary: $\Psi_{ext} \propto (1+e^{i\delta})\Psi_{in}$

$e^{i\delta}$ is reflection amplitude of Fresnel equations.



⇐ Example.

DBC better for modes with large m/n (or θ_i) and much better for TE than TM polarization.


Florianópolis
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PART II. "Phase shift" boundary conditions

- For TM, TE polarizations, use the correct Fresnel reflection amplitudes $r=e^{i\delta(\theta_i)}$.

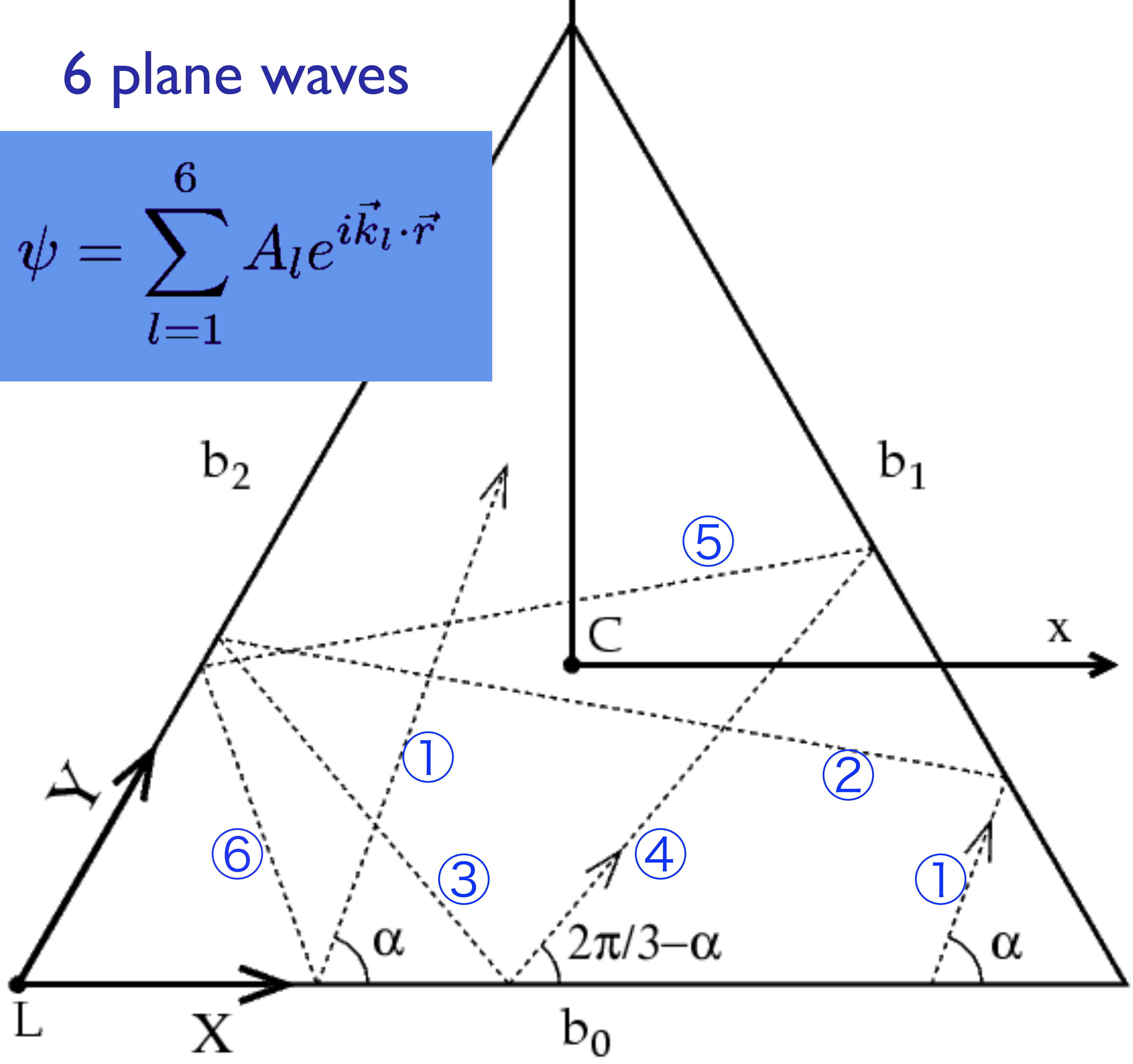
- (Dirichlet BC -- reflection phase shift is $\delta=-\pi$.)
(Neumann BC -- reflection phase shift is $\delta=0$.)

- Fields generally are not zero at the boundary. There are evanescent waves outside the cavity.

- Try to get nearly correct field matching from inside to outside the cavity.

6 plane waves

$$\psi = \sum_{l=1}^{6} A_l e^{i\vec{k}_l \cdot \vec{r}}$$



Main Objective:

Find allowed k, α.

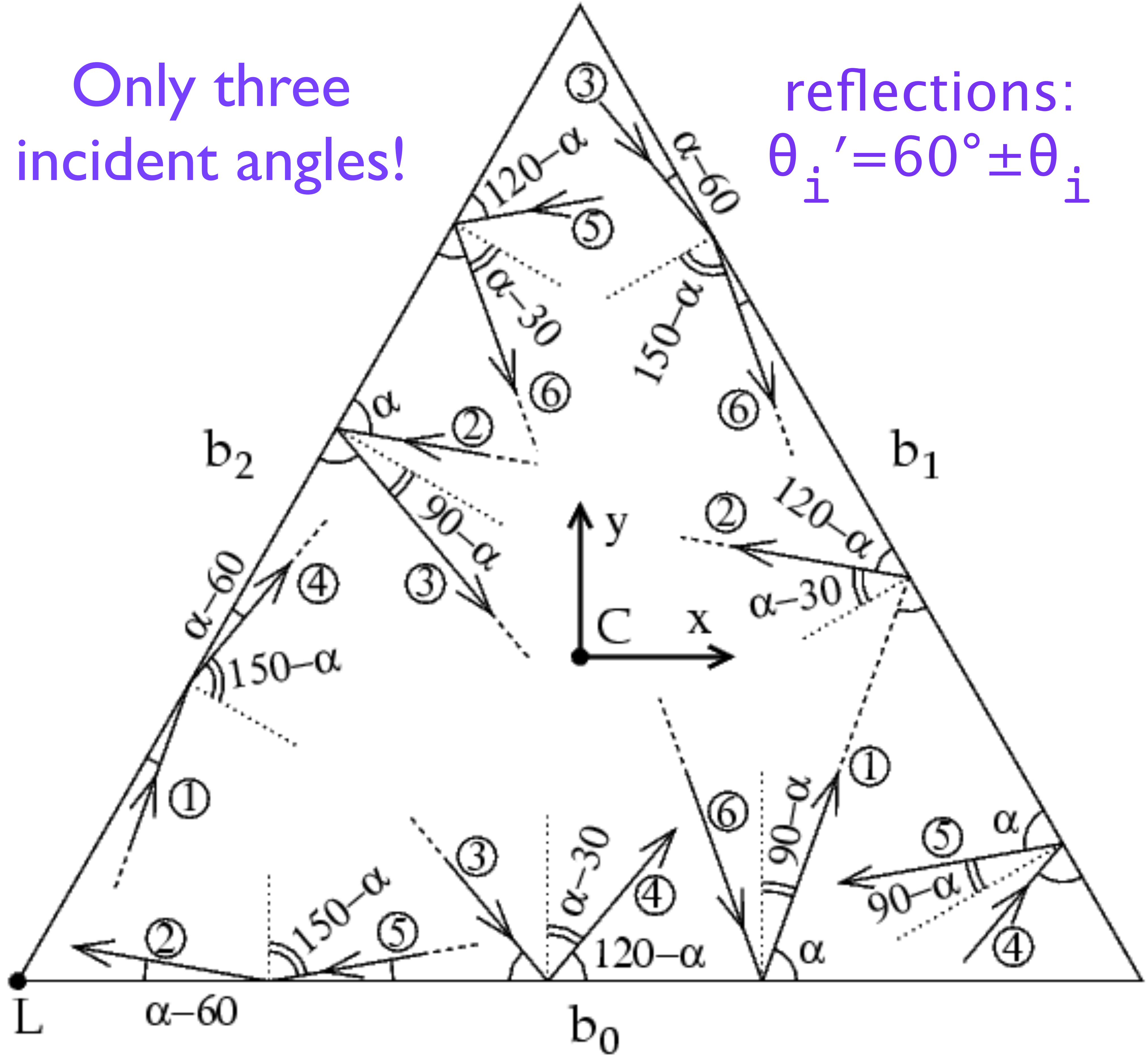
Assume $\alpha > 60°$

$$k_1 = k(\cos\alpha, \sin\alpha) = (k_x, k_y).$$
$$k_6 = k(\cos\alpha, -\sin\alpha) = (k_x, -k_y).$$

etc....

Only three incident angles!

reflections: $\theta_i' = 60° \pm \theta_i$

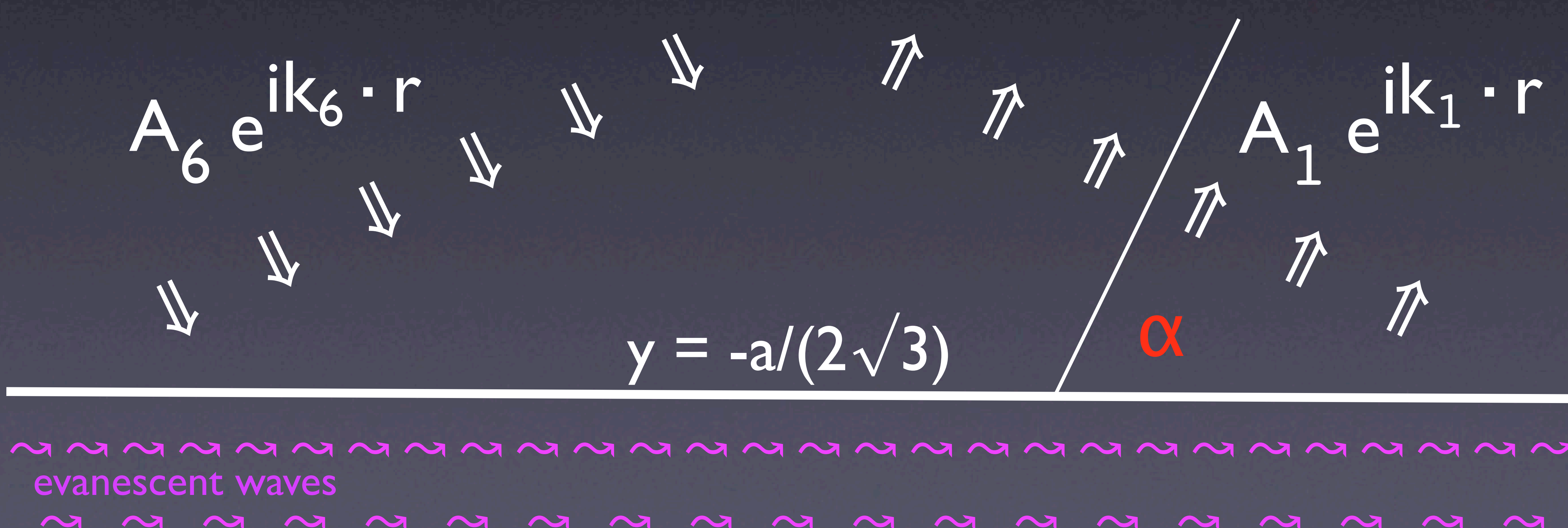


matching waves ⑥ and ① at lower boundary

Fresnel! coefficient

$$r_6 = e^{i\delta_6} = \frac{A_1 e^{-ik_y a/2\sqrt{3}}}{A_6 e^{ik_y a/2\sqrt{3}}}$$

$$k_y = k_{1y} = k \sin \alpha$$



net phase shifts

wave ⑥ reflects to generate wave ①

$$A_1 = A_6\, e^{i\Delta_6}$$

$$\boxed{\Delta_6 \equiv \delta_6 - k_{6y}\, a/\sqrt{3}}$$

(refl.= inc. * $e^{i\Delta_6}$)

$$k_{6y} = -k \sin\alpha$$

similar for incident waves ③ and ⑤ on lower boundary...

$$A_4 = A_3\, e^{i\Delta_3}$$

$$\boxed{\Delta_3 \equiv \delta_3 - k_{3y}\, a/\sqrt{3}}$$

$$A_2 = A_5\, e^{i\Delta_5}$$

$$\boxed{\Delta_5 \equiv \delta_5 - k_{5y}\, a/\sqrt{3}}$$

Use symmetry on other boundaries, leads to a nice problem:

$$\frac{A_4}{A_3} = \frac{A_2}{A_1} = \frac{A_6}{A_5} = e^{i\Delta_3},$$

$$\frac{A_2}{A_5} = \frac{A_6}{A_3} = \frac{A_4}{A_1} = e^{i\Delta_5},$$

$$\frac{A_1}{A_6} = \frac{A_5}{A_4} = \frac{A_3}{A_2} = e^{i\Delta_6},$$

Follow sequences of internal reflections of the waves

1 ⇨ 2 ⇨ 3 ⇨ 4 ⇨ 5 ⇨ 6 ⇨ 1,

$$A_1 \frac{A_2}{A_1} \frac{A_3}{A_2} \frac{A_4}{A_3} \frac{A_5}{A_4} \frac{A_6}{A_5} \frac{A_1}{A_6} = A_1$$

$$\boxed{e^{i \cdot 3(\Delta_3+\Delta_6)} = 1}$$

1 ⇨ 4 ⇨ 5 ⇨ 2 ⇨ 3 ⇨ 6 ⇨ 1,

$$A_1 \frac{A_4}{A_1} \frac{A_5}{A_4} \frac{A_2}{A_5} \frac{A_3}{A_2} \frac{A_6}{A_3} \frac{A_1}{A_6} = A_1$$

$$\boxed{e^{i \cdot 3(\Delta_5+\Delta_6)} = 1}$$

1 ⇨ 2 ⇨ 3 ⇨ 6 ⇨ 1,

$$A_1 \frac{A_2}{A_1} \frac{A_3}{A_2} \frac{A_6}{A_3} \frac{A_1}{A_6} = A_1$$

$$\boxed{e^{i \cdot (\Delta_3+\Delta_5+2\Delta_6)} = 1}$$

Mode condition: The exponents must be 2π * integers!

Another way to look at it.

evens → odds →

$$\frac{A_4}{A_3} = \frac{A_2}{A_1} = \frac{A_6}{A_5} = e^{i\Delta_3},$$

evens → odds →

$$\frac{A_2}{A_5} = \frac{A_6}{A_3} = \frac{A_4}{A_1} = e^{i\Delta_5},$$

odds → evens →

$$\frac{A_1}{A_6} = \frac{A_5}{A_4} = \frac{A_3}{A_2} = e^{i\Delta_6},$$

Quantization. n_3, n_5, n_6 = integers

$$\boxed{\begin{aligned} 3(\Delta_3+\Delta_6)&=2\pi n_3 \\ 3(\Delta_5+\Delta_6)&=2\pi n_5 \\ \Delta_3+\Delta_5+2\Delta_6&=2\pi n_6 \end{aligned}}$$

But! $(n_3+n_5)=3n_6$

$\delta_3,\delta_5,\delta_6$ depend on α

Apply defns. of the Δ's, then,

$$\boxed{\begin{aligned} k_x a &= ka\cos\alpha = (\Delta_3-\Delta_5) - (\delta_3-\delta_5) \\ 3^{1/2}\, k_y a &= 3^{1/2}\, ka\sin\alpha = (\Delta_3+\Delta_5+2\Delta_6)-(\delta_3+\delta_5+2\delta_6) \end{aligned}}$$

Can solve for α by eliminating k...
Gives nonlinear equation for α.

Fresnel phase shifts $\delta(\theta_i)$

TM=perpendicular polarization

$$\frac{\psi_{\text{refl.}}}{\psi_{\text{inc.}}} = e^{i\delta} = \frac{\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta'}{\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta'}$$

$$\tan\frac{\delta}{2} = -\frac{\mu}{\mu'}\sqrt{\frac{\cos^2\theta_c}{\cos^2\theta_i} - 1} \qquad \text{(TM)}.$$

$$\cos\theta' = i\gamma' \equiv i\sqrt{(\sin\theta_i/\sin\theta_c)^2 - 1},$$

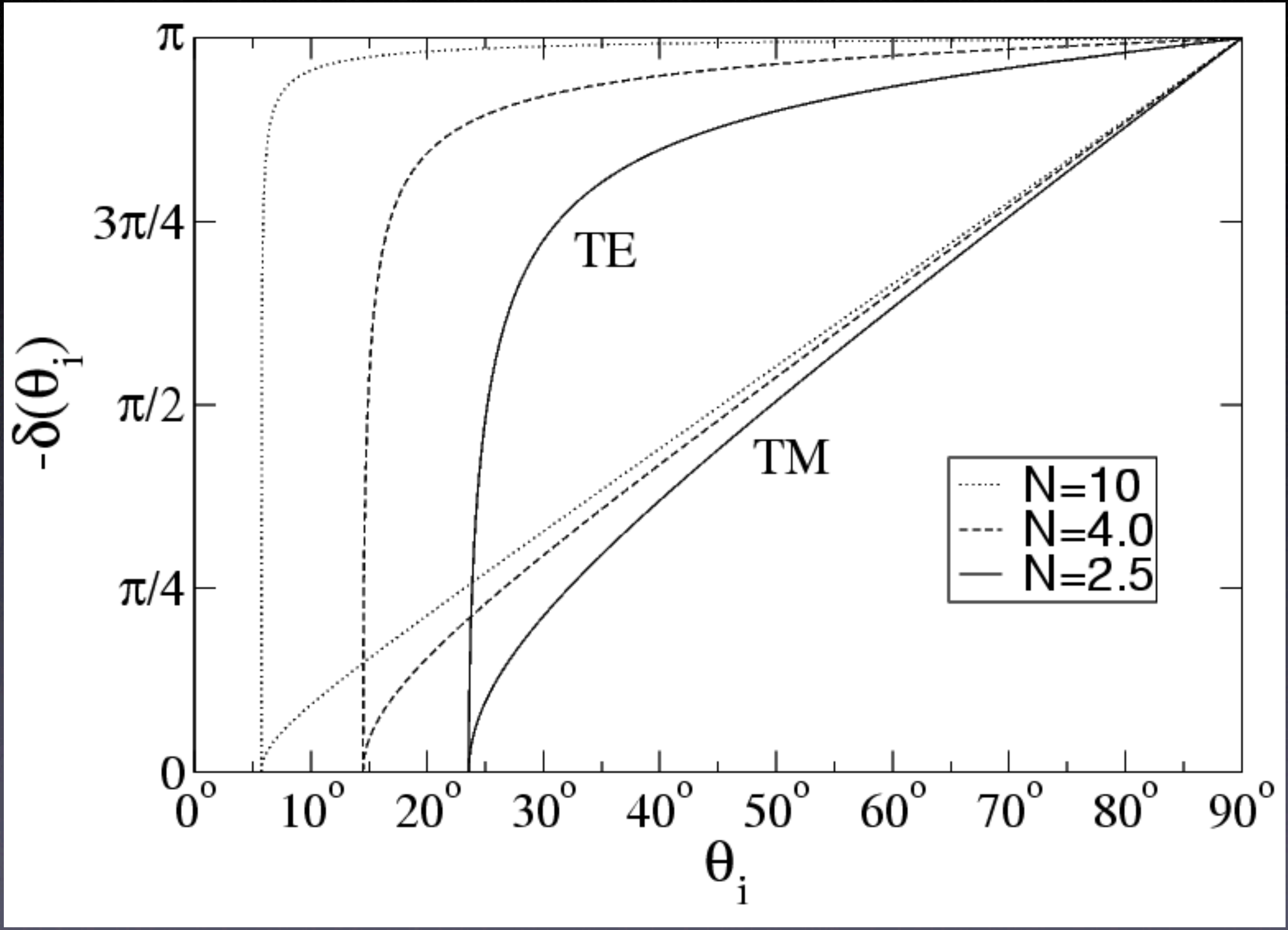
TE=parallel polarization

$$\frac{\psi_{\text{refl.}}}{\psi_{\text{inc.}}} = e^{i\delta} = \frac{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i - \sqrt{\frac{\epsilon}{\mu}}\cos\theta'}{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i + \sqrt{\frac{\epsilon}{\mu}}\cos\theta'}$$

$$\tan\frac{\delta}{2} = -\frac{\epsilon}{\epsilon'}\sqrt{\frac{\cos^2\theta_c}{\cos^2\theta_i} - 1} \qquad \text{(TE)}.$$

stronger index dependence!

TE phase shifts are closer to -π



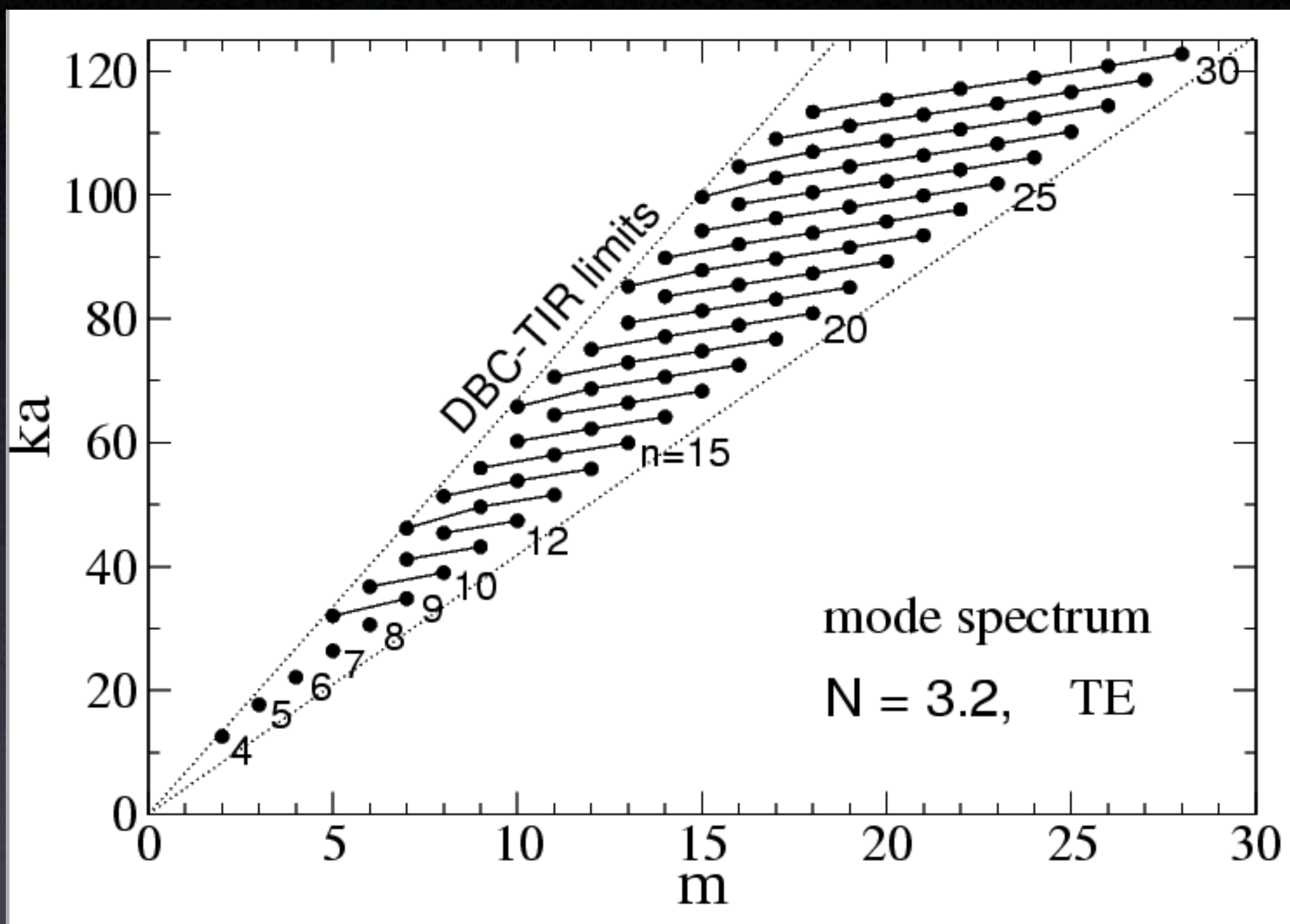
spectrum

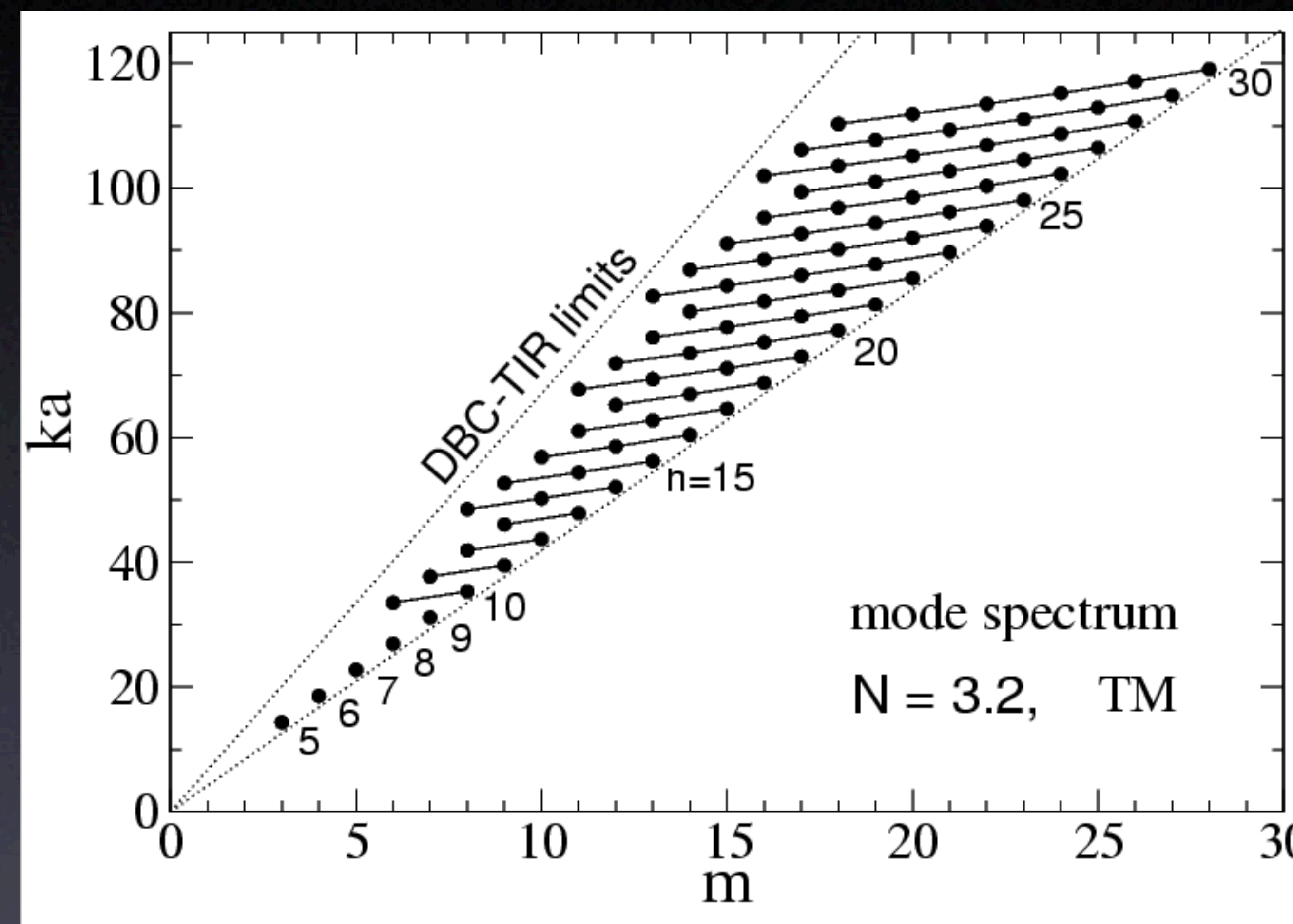
$ka=(\omega/c)(\varepsilon\mu)^{1/2}$

$$m \equiv (n_3 - n_5), \qquad n \equiv (n_6 + 2).$$

TE

TM





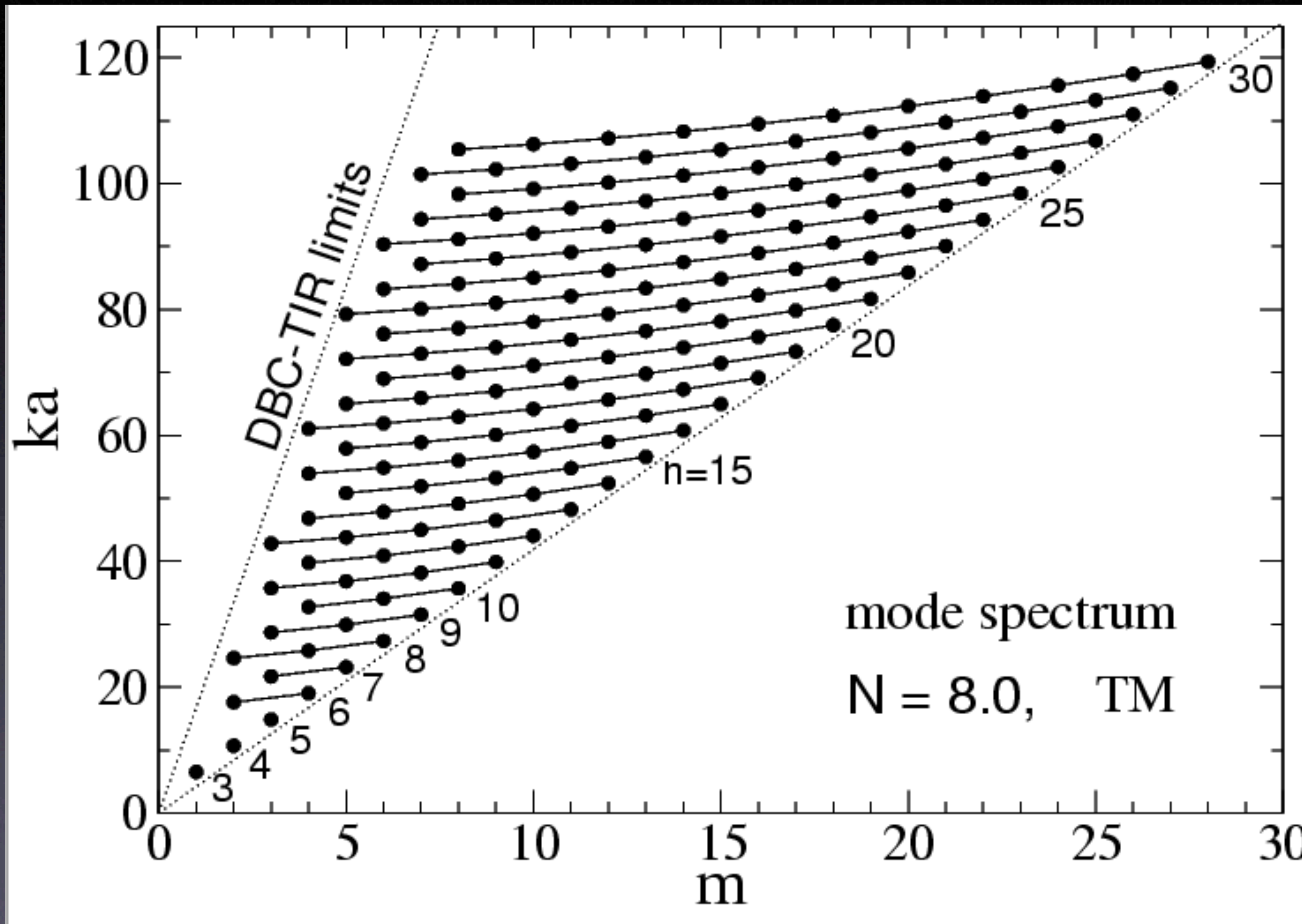
all modes within DBC- TIR limits:

$4\pi m/3 < ka < (4\pi m/3)(N/2)$

$(m=n)$ $(\alpha=60°)$

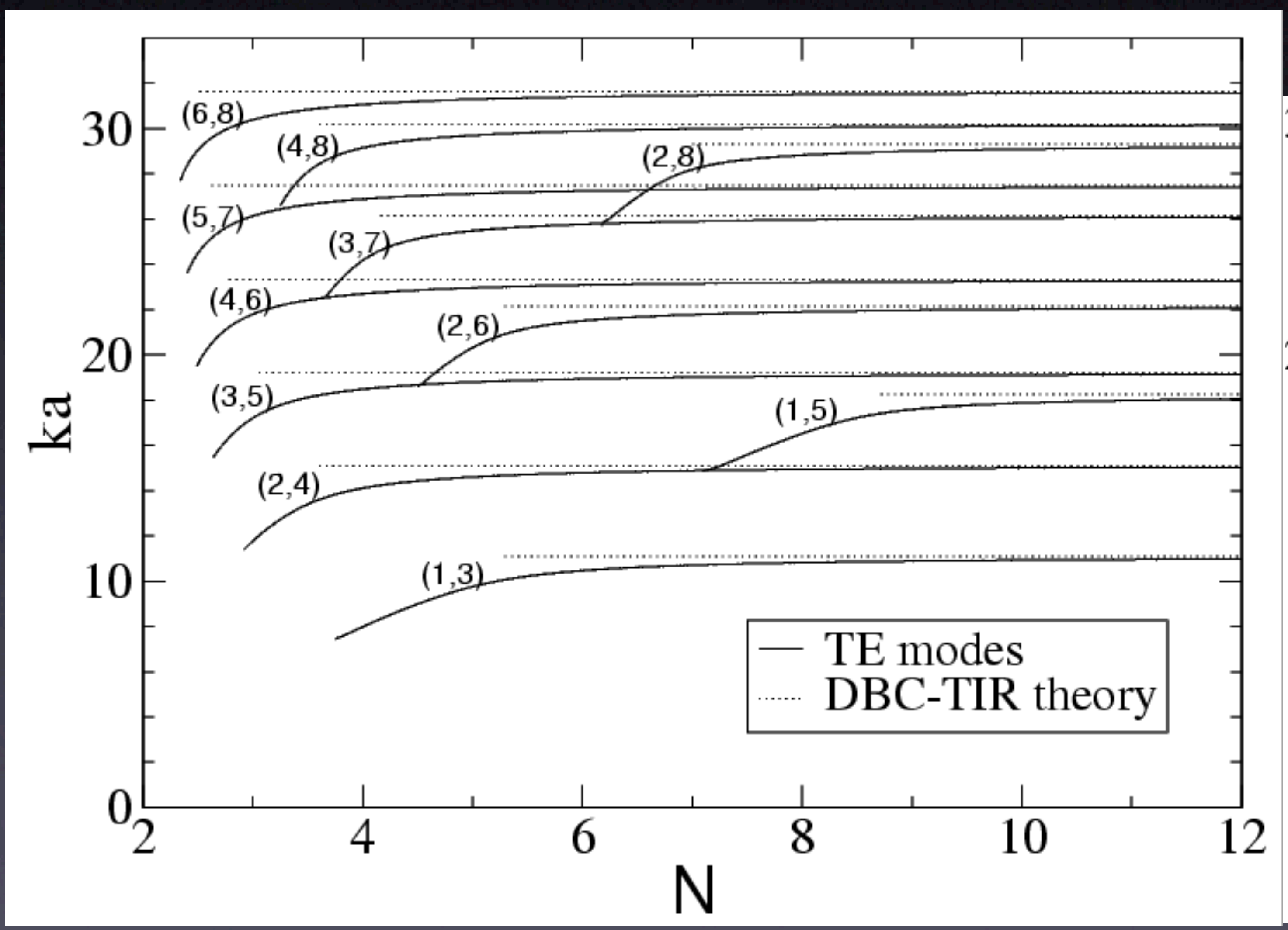
$(\sin\theta_i=1/N)$

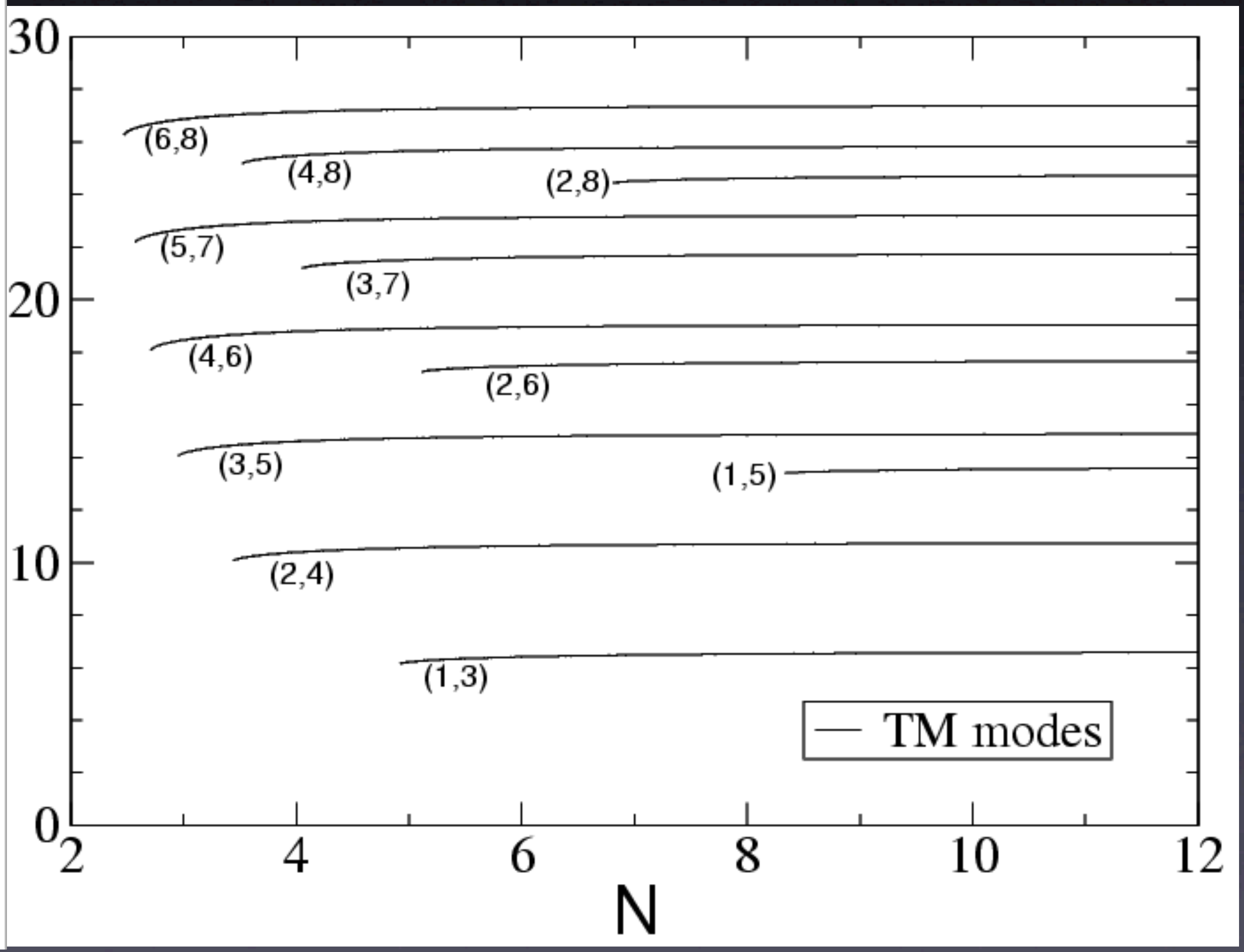
larger N → more modes.



TE almost follow DBC theory.

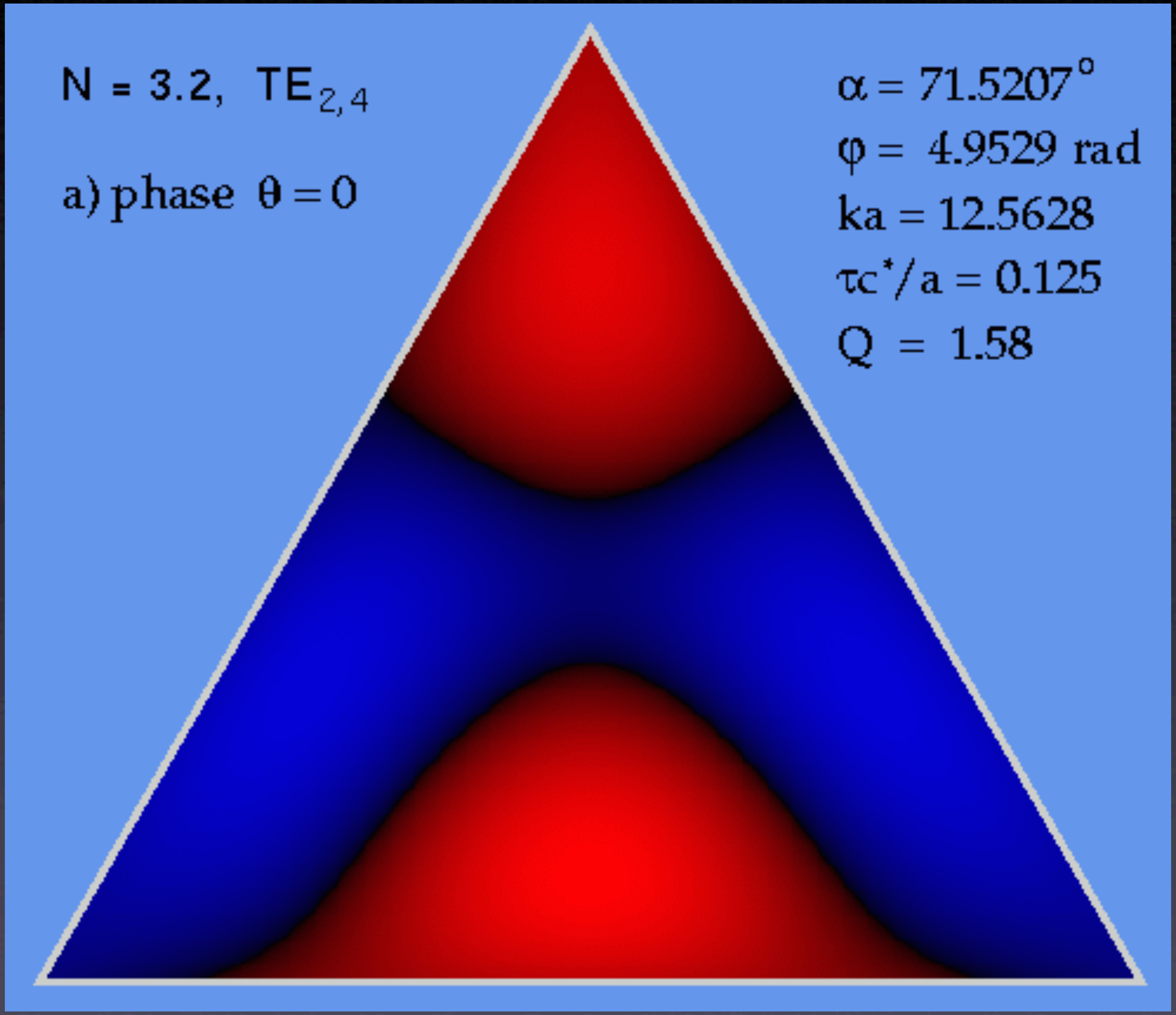
TM fall at lower frequencies.

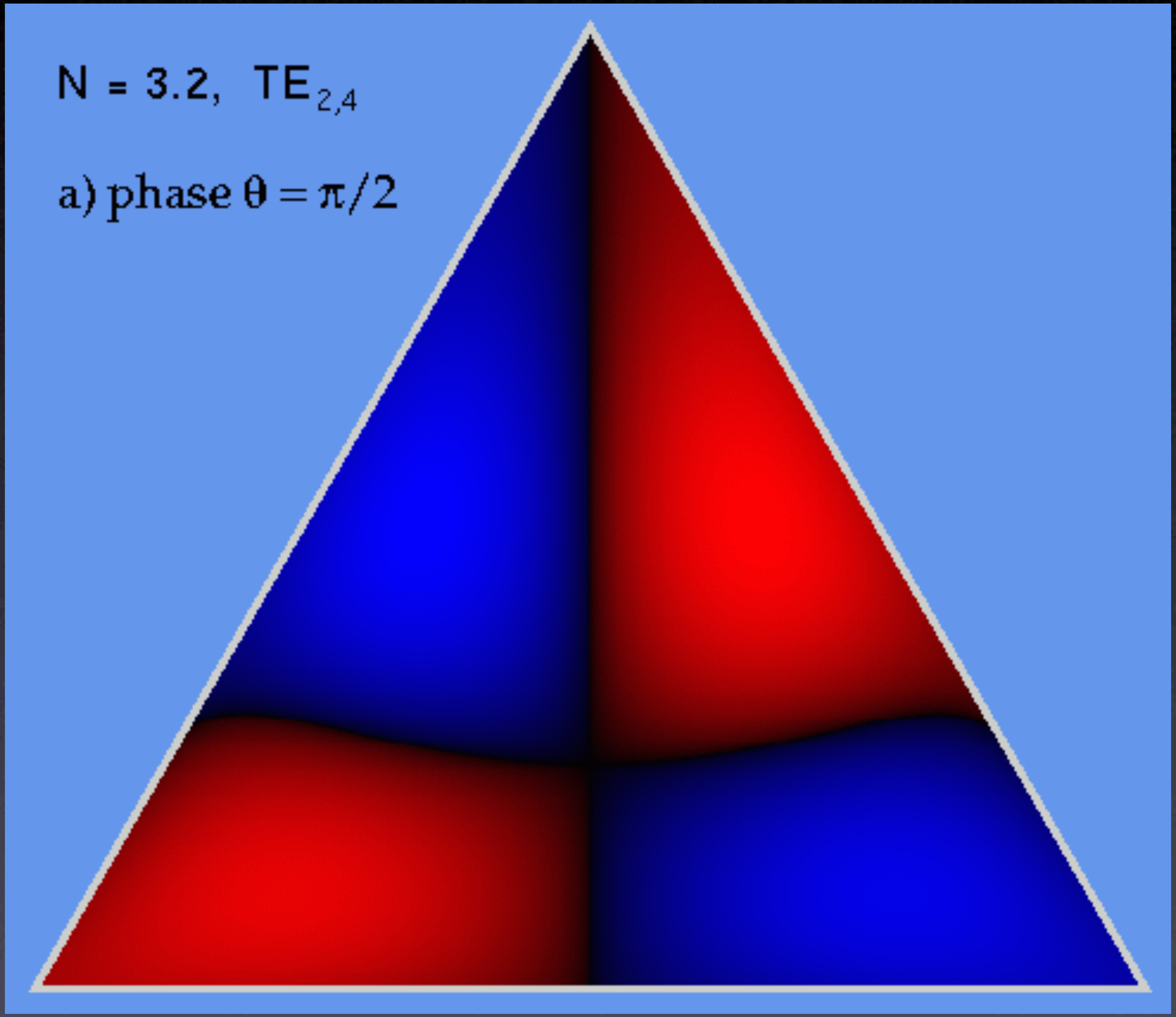




N=n'/n=index ratio

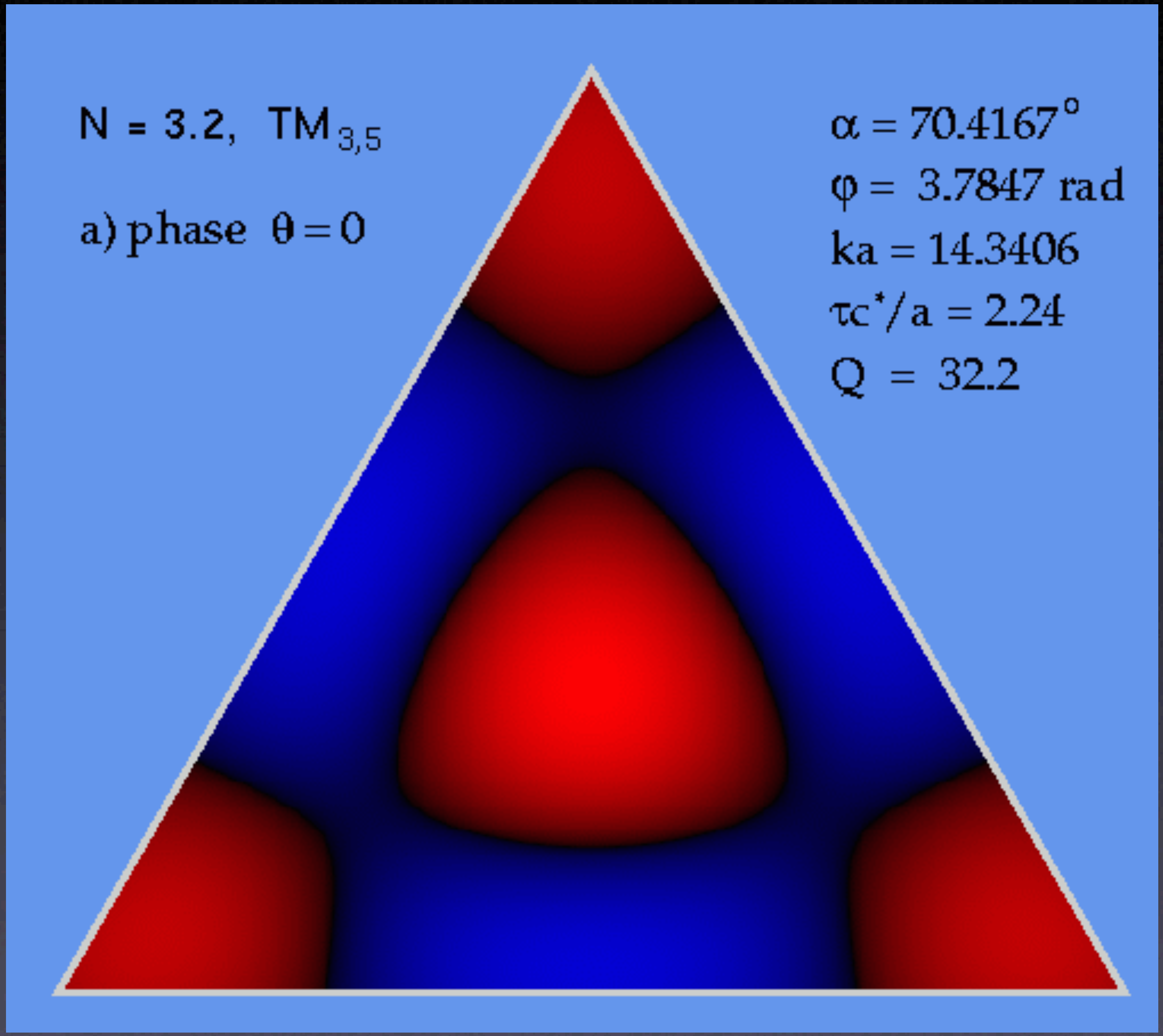
lowest mode at N=3.2 (doubly degenerate) $TE_{2,4}$

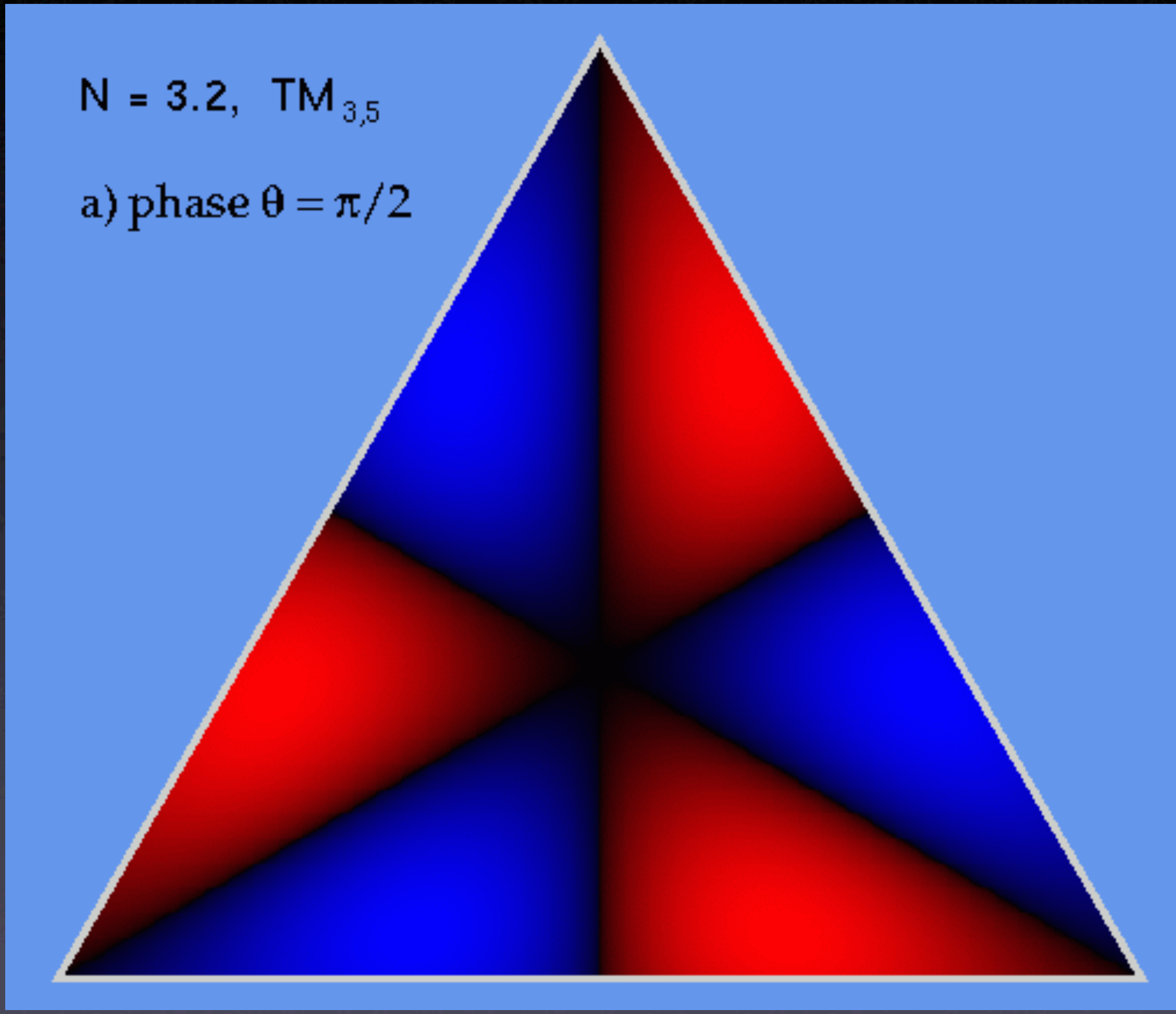




1st excited mode at N=3.2

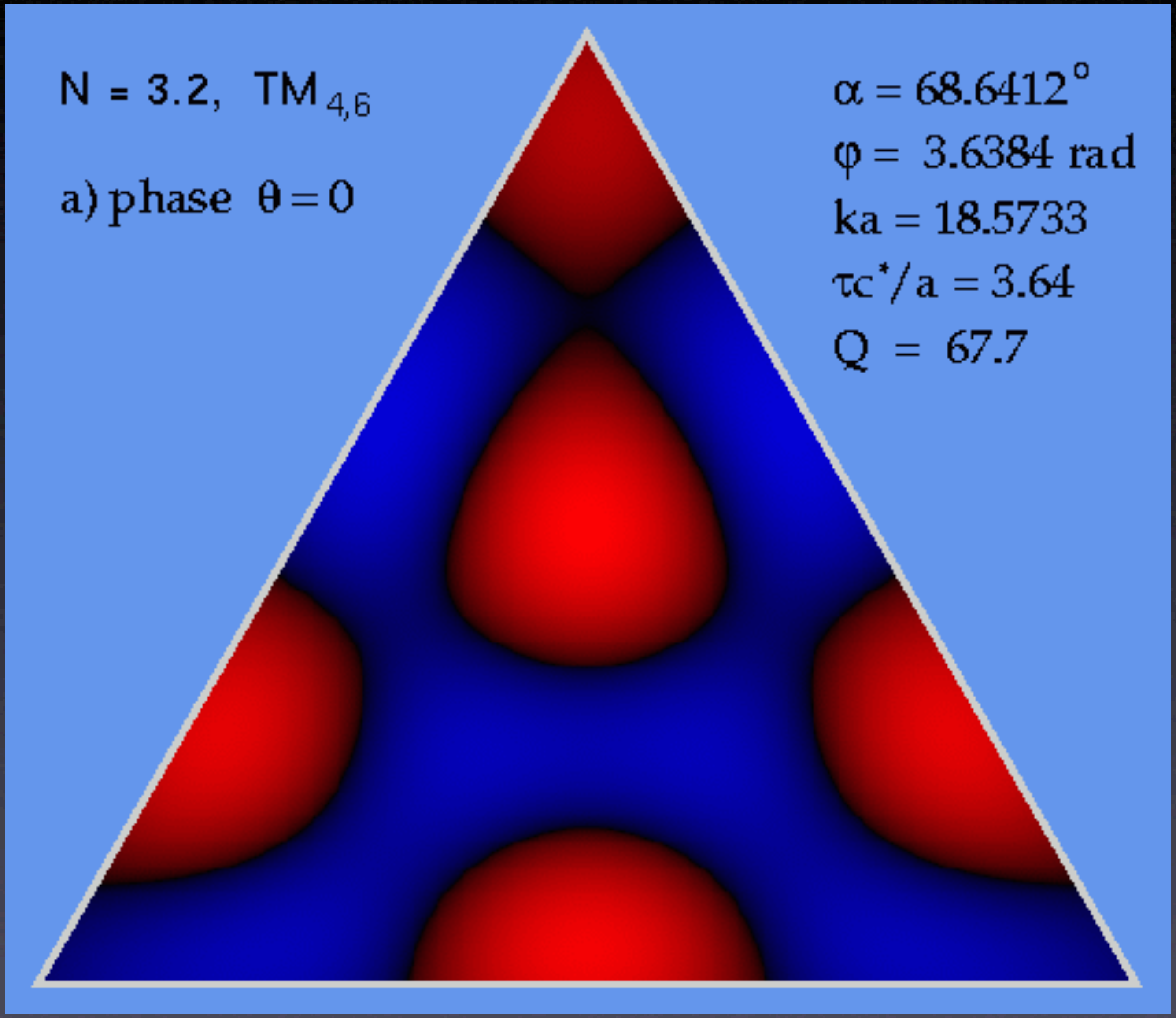
(doubly degenerate) $TM_{3,5}$

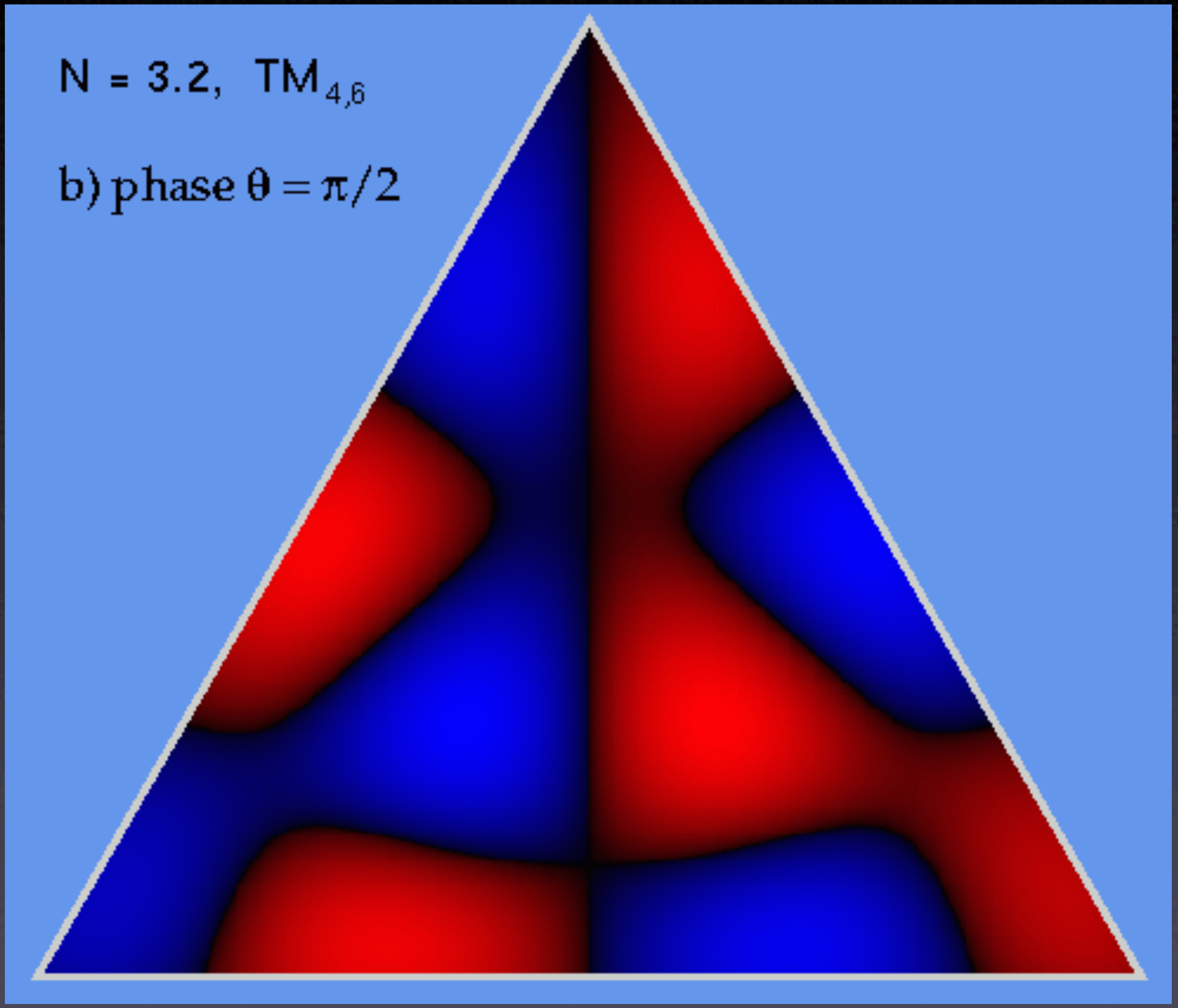




another excited mode at N=3.2

(doubly degenerate) $TM_{4,6}$



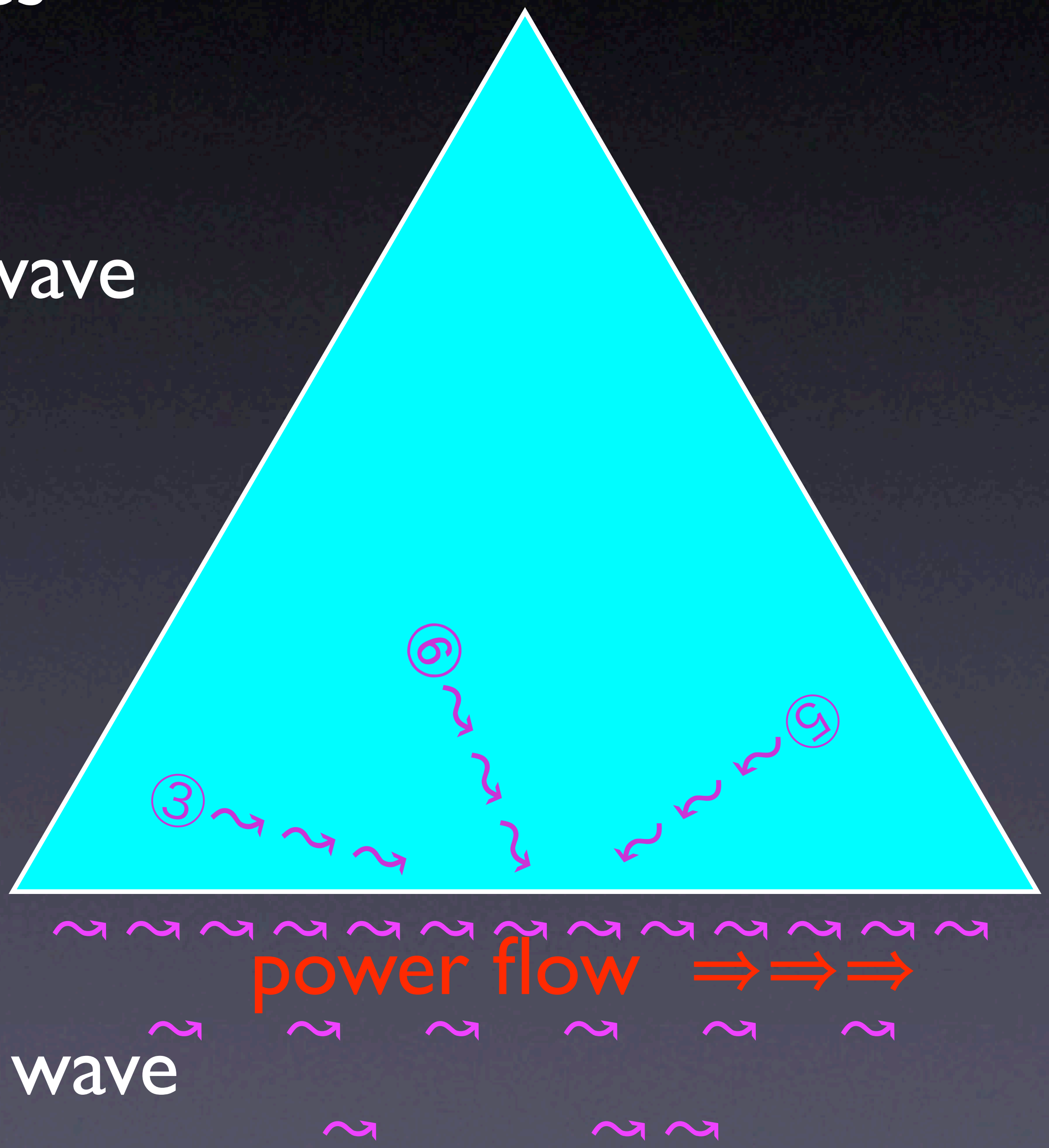


Mode lifetime estimates--due to escape of evanescent boundary waves at the triangle vertices (Wiersig 2003).

Solution has three plane waves incident on any boundary.

Wave ⑥ produces evanescent wave of greatest power (min. θ_i)

$$\boxed{\text{lifetime } \tau \approx U_{cavity} \div 3P_{⑥}}$$

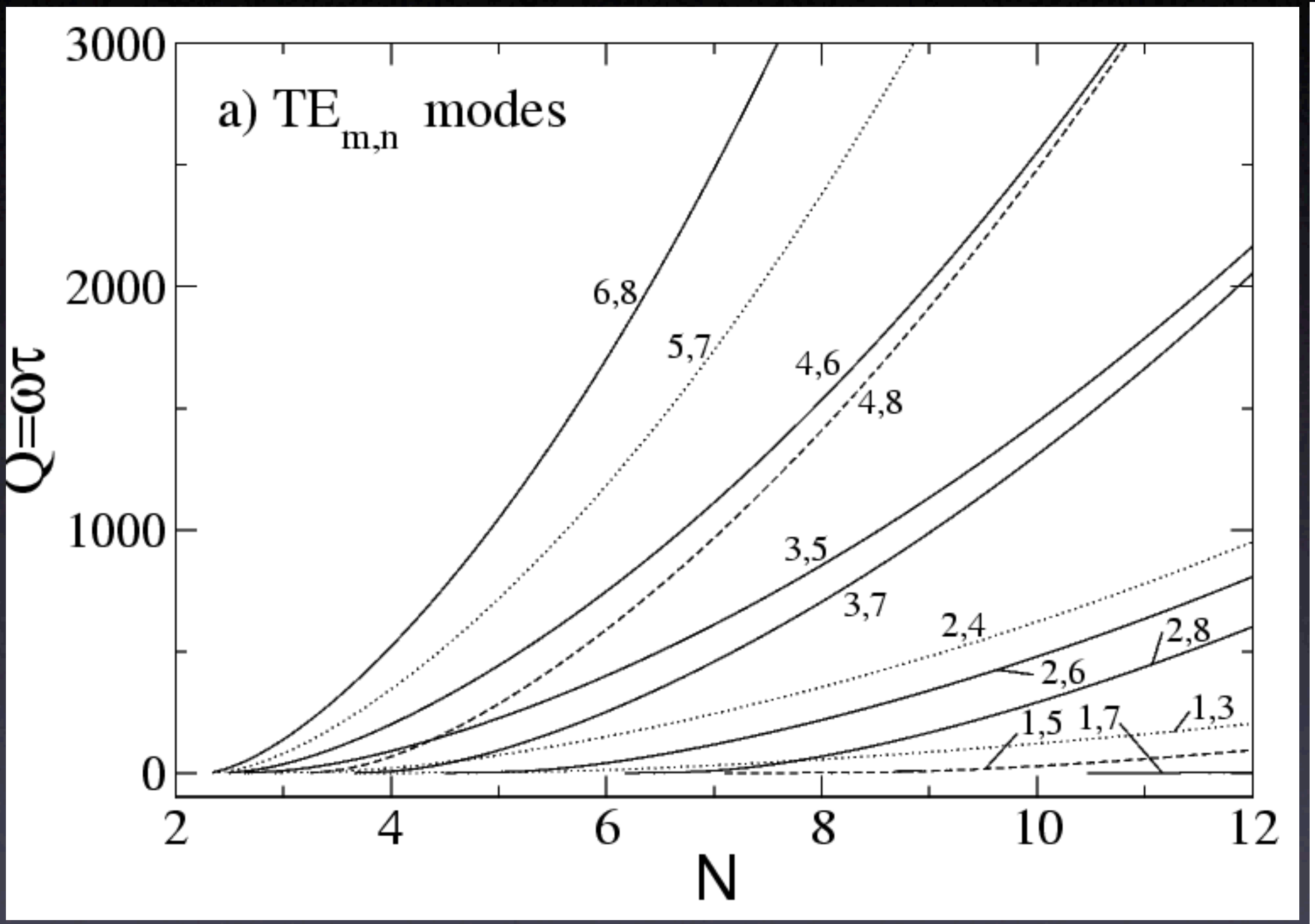


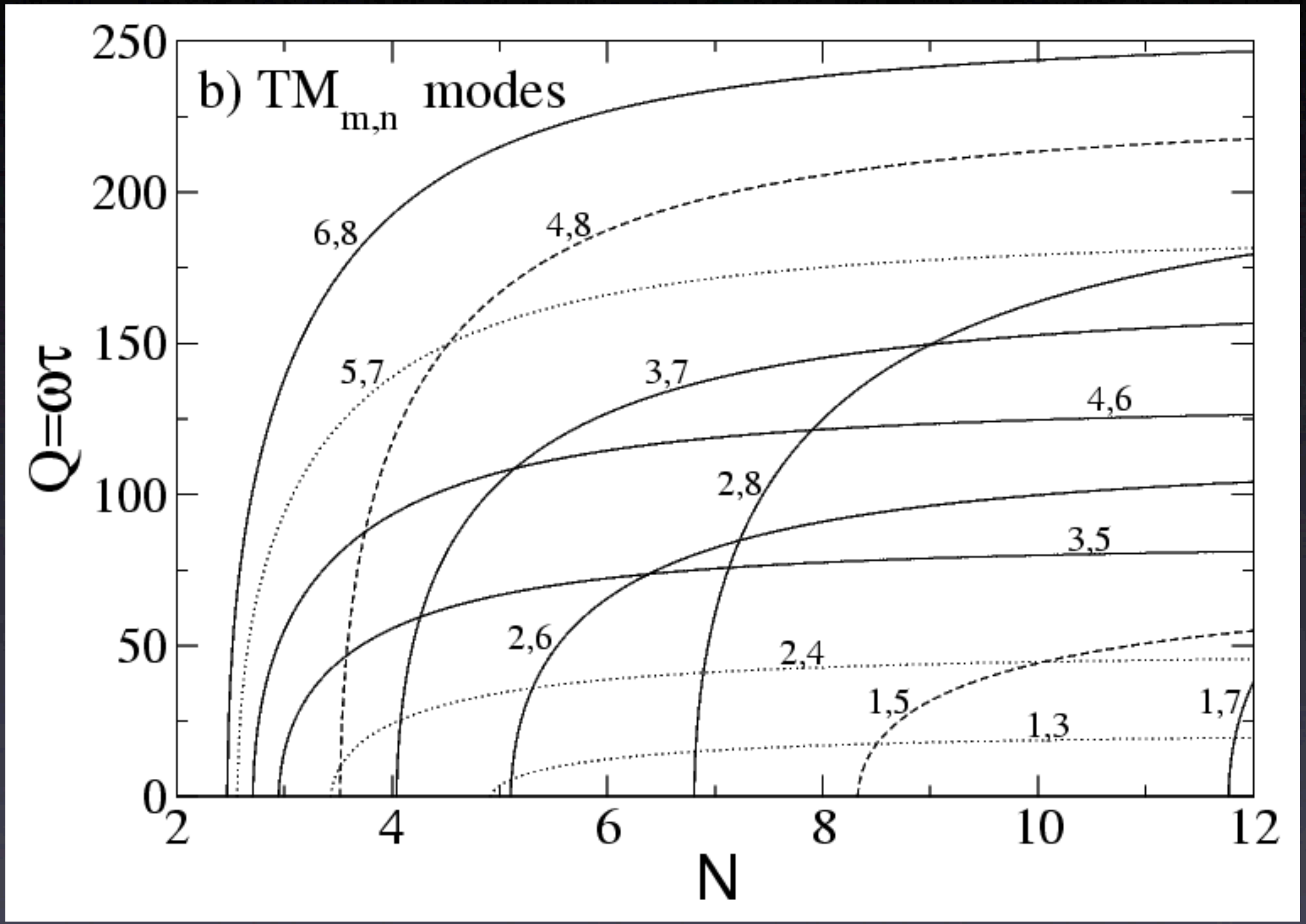
U_{cavity} = total cavity energy

$P_{⑥}$ = power in ⑥'s evanescent wave

mode quality factors vs. index ratio N=n/n'

Q=ωτ

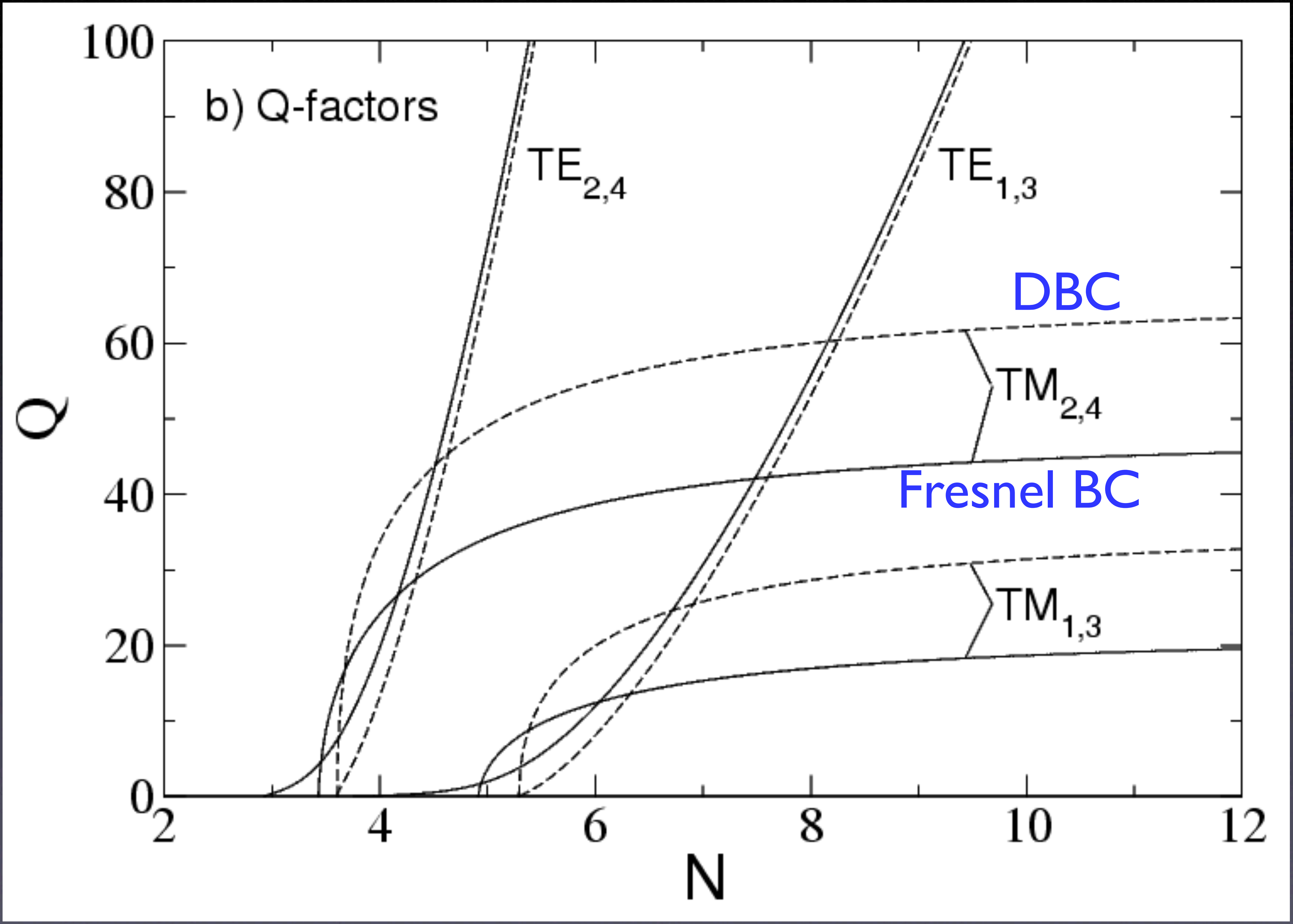




$$\tau_{TE} \sim (\varepsilon/\varepsilon')(\varepsilon\mu)^{1/2}(a/c)$$

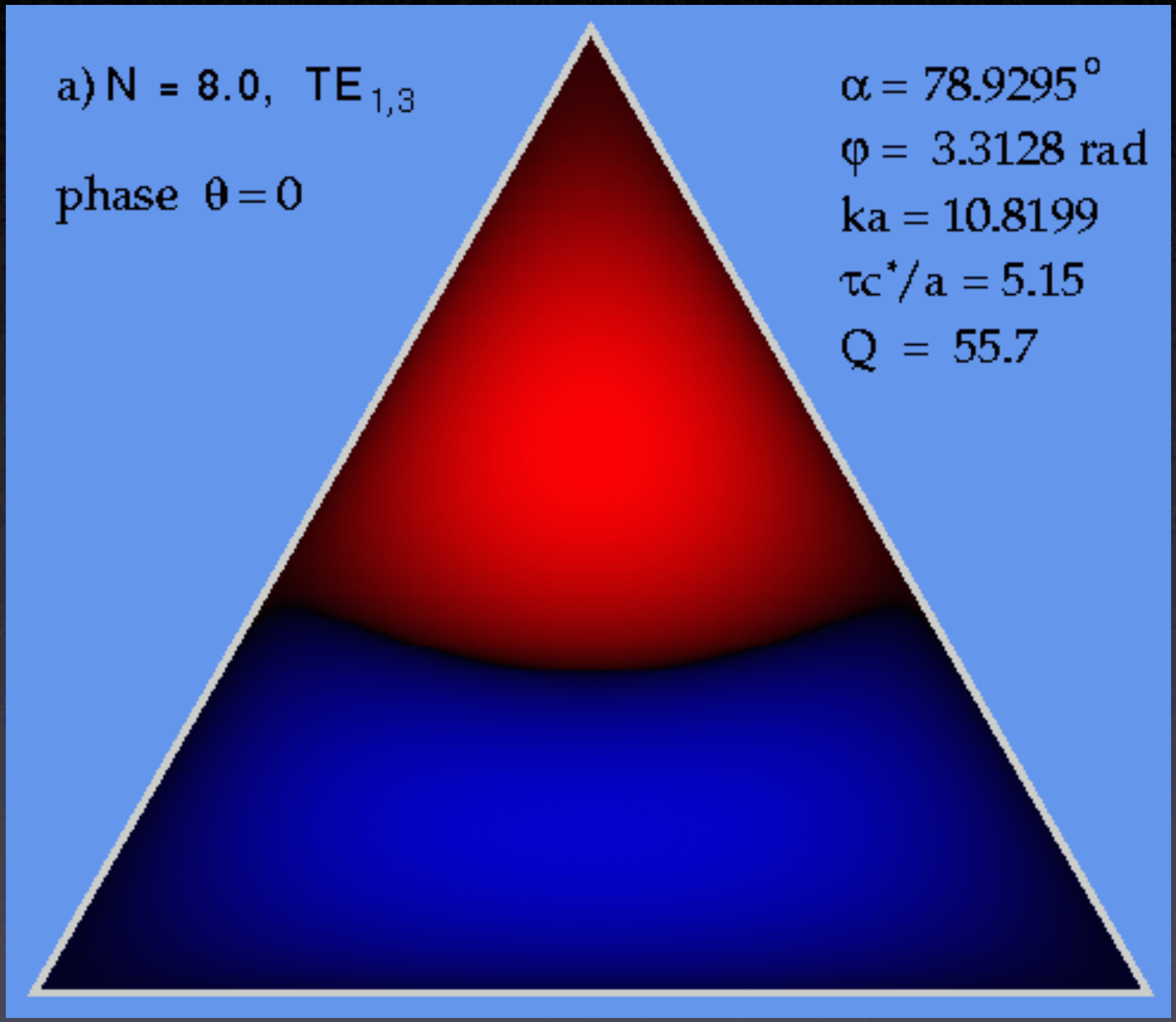
$$\tau_{TM} \sim (\varepsilon\mu)^{1/2}(a/c)$$

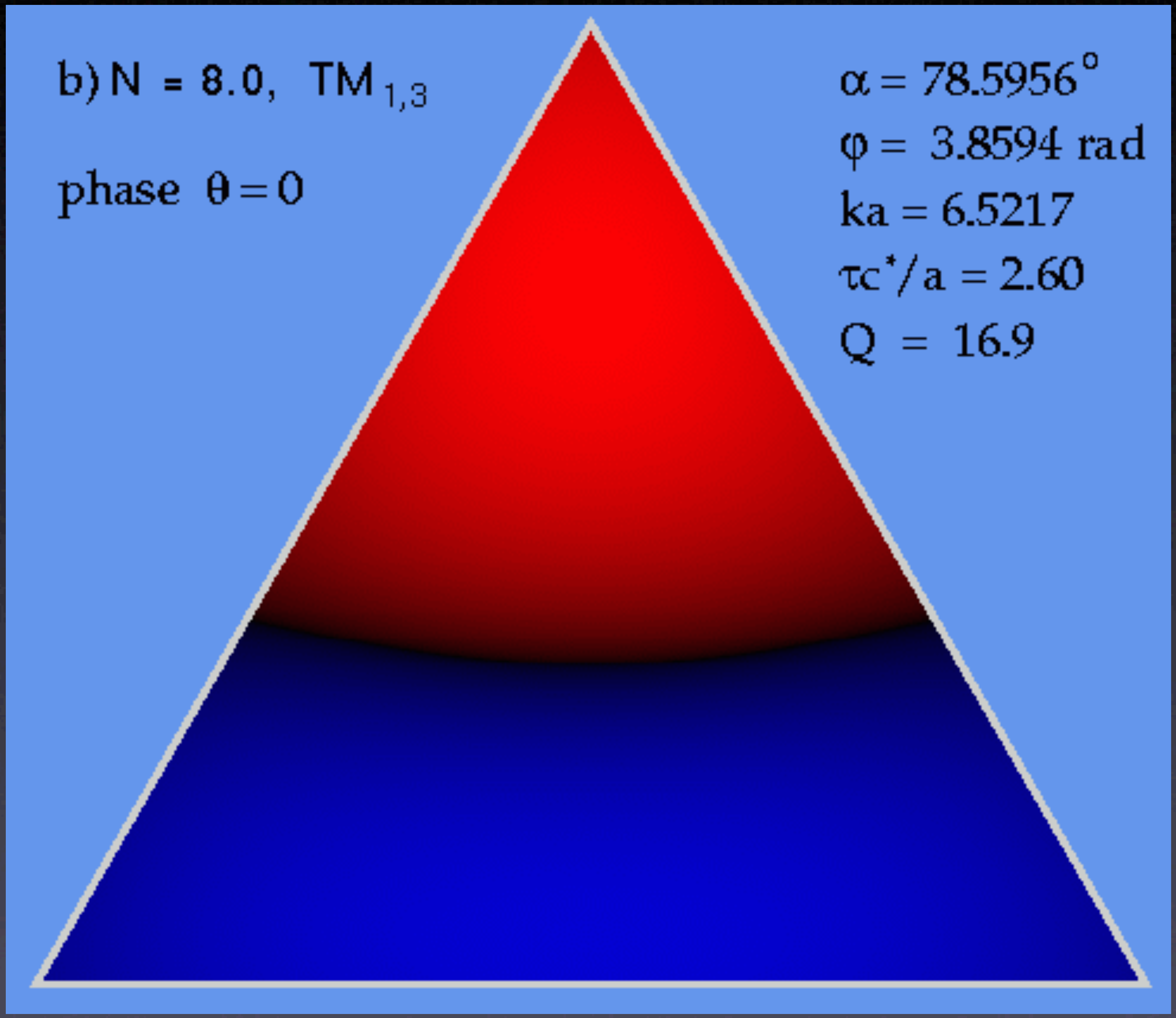
comparison of Q-factors vs. index ratio N=n/n'



N=8.0

$TE_{1,3}$ $TM_{1,3}$





(more confined)

Errors? in $Q=\omega\tau$ calculation ?

- ? Do the evanescent waves react back on the waves inside the cavity ? How do they really scatter at the corners ?
- Y.Z. Huang et al. used finite-difference-time-domain + Padé approx. technique to study scattering of incident waves by a triangle.
  $\Rightarrow$ contrary results: larger Q's for TM modes.

Conclusions

- Used Fresnel reflection coefficients $r=e^{i\delta}$ to phase match the 6 plane waves inside the ETR.
- Solved this eigenvalue problem to get ray-angle α, then k, ω, Ψ, for TE and TM polarizations.
- Plane wave components leak out of cavity via evanescent boundary waves, leading to finite lifetimes τ.
- At large N, TE modes are found to have longer lifetimes and higher Q's than TM modes.

EECE Distinguished Lecture

The Life and Times of James Clerk Maxwell

Dr. James C. Rautio

2005 IEEE Distinguished Lecturer

Friday, 21 October 2005

3:30 p.m.

Fiedler Auditorium

In his 90-minute presentation, Dr. Rautio covers the life of Dr. James Clerk Maxwell from the viewpoint of a microwave engineer, drawing on many sources in providing an understanding of James Maxwell himself. In many ways, James Clerk Maxwell stands shoulder to shoulder with Newton and Einstein, yet even those of us who have spent decades working with Maxwell's equations are almost totally unfamiliar with his life and times. What was Maxwell like as an infant? What was the tragedy at eight years old that profoundly influenced his life? What unique means of transportation did young Maxwell use to escape a cruel tutor? What memorable event occurred on his first day of school? When did he publish his first papers, and what were they about? What did Maxwell have to do with the rings of Saturn? Why did he lose his job as a professor? Why did he have a hard time getting another job? What was his wife like? What is Maxwell's legacy to us? The answers to these questions provide insight into Maxwell the person and add an extra dimension to those four simple equations we have studied ever since.

Dr. James C. Rautio is President and Founder of Sonnet Software, Inc., a firm offering software for analysis and synthesis involving high-frequency electromagnetics. A Fellow of the Institute of Electrical and Electronics Engineers (IEEE), he was selected by the Microwave Theory and Techniques Society of the IEEE as a 2005 Distinguished Microwave Lecturer. Dr. Rautio's visit to our area is at the invitation of The Consultants Network of the Kansas City Section of the IEEE, and his presentation at K-State is part of the Department of Electrical and Computer Engineering's Distinguished Lecture Program.