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MASS AND MOMENTUM FOR VORTICES IN TWO-DIMENSIONAL EASY-PLANE MAGNETS

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INTRODUCTION

Many quasi-two-dimensional magnetic systems are expected to allow particle-like vortex excitations that are spin configurations with a net 2π twist about a particular point or core. Examples are XY or easy-plane quasi-2d magnets such as $BaCoAsO_4$, and the nearly Heisenberg-like K_2CuF_4 with a weak easy-plane anisotropy.[1] Vortices are also possible in systems such as monolayer magnetic lipid systems.[2] Vortices are created or destroyed in particle-antiparticle pairs, carry a conserved circulation charge, exert pair interaction forces on each other, and contribute to thermodynamics and spin-correlation functions.[3] There has been much interest in their role in a topological transition due to vortex-antivortex unbinding[4] above a characteristic temperature T_{KT}.

The vortex contributions to spin-correlations can be calculated approximately[5,6] by assuming a gas of weakly interacting vortices for temperatures above T_{KT}, with a Boltzmann velocity distribution characterized by an average thermal speed v_{rms}. The thermal speed was estimated by Huber from a velocity autocorrelation function based on a vortex equation of motion introduced by Thiele.[7] However, there are actually two different types of vortices possible in the easy-plane magnet, known as "in-plane" and "out-of-plane", depending on whether the out-of-easy-plane spin component is zero or nonzero for the stationary vortex.[8,9] The Thiele equation has been found to be inadequate to describe the in-plane vortices. A related point is that not all of the dynamic properties of magnetic vortices are fully understood, especially concerning the concept of vortex momentum.

For these reasons a new equation of motion for vortices has been proposed.[10] The principal physical effect included in the new equation is that a moving vortex can possess a mass. The spin profile for a moving vortex depends on the velocity. For example, the out-of-plane spin components increase with velocity, and this is responsible for the mass. A momentum can also be associated with the mass, and these carry consequences for the motion of interacting pairs of vortices.

Vortex dynamics is also determined by a second type of charge known as the gyrovector. The total gyrovector of the system is conserved for a continuum limit, but we find that it is not conserved when generalized to a discrete lattice, unlike the circulation charge. As a result vortex-antivortex annihilation can occur in a lattice system when it would be prohibited in the continuum system.

The new equation of motion with mass was discussed previously,[10] starting from a definition of momentum. Here we take an alternative approach where momentum need not be defined in order to obtain the dynamic equation, starting from the Landau-Lifshitz equation for the spin dynamics.[11] This will be followed by some discussion of the vortex momentum and problems with its definition. These results will be related to predicting motions of interacting pairs of vortices, as well as how to determine vortex masses from simulations. Simulations show that the gyrovector is not a conserved quantity for lattice systems. We begin by summarizing some of the properties of in-plane and out-of-plane vortices in the easy-plane ferromagnet.

2D EASY-PLANE FERROMAGNET AND VORTICES

We consider a Heisenberg model with ferromagnetic exchange $J > 0$ and easy-plane anisotropy characterized by $0 < \delta \leq 1$, with Hamiltonian

$$H = -J \sum_{(\mathbf{n},\mathbf{m})} (\vec{S}_{\mathbf{n}} \cdot \vec{S}_{\mathbf{m}} - \delta S^z_{\mathbf{n}} S^z_{\mathbf{m}}). \tag{1}$$

$\vec{S}_{\mathbf{n}}$ is a classical 3d spin vector at site $\mathbf{n}$ in a 2D square lattice, and the sum is over near-neighbor bonds. The limits $\delta = 0$ and $\delta = 1$ correspond to the isotropic Heisenberg and XY models, respectively. The individual spin length S is conserved, and the dynamic variables are the in-plane angle $\phi_{\mathbf{n}} = \tan^{-1}(S^y_{\mathbf{n}}/S^x_{\mathbf{n}})$ and its canonically conjugate momentum $S^z_{\mathbf{n}}$.

In a continuum limit, the lattice point $\mathbf{n}$ goes over into position $\vec{x}$, and $\vec{S}(\vec{x},t)$ has dynamics given from Poisson brackets,

$$\frac{d\vec{S}}{dt} = \{\vec{S}, H\} = \vec{S} \times \vec{h}, \tag{2a}$$

$$\vec{h} = -\frac{\delta H}{\delta \vec{S}} = \nabla^2 \vec{S} - \delta(4S^z + \nabla^2 S^z)\hat{e}_z. \tag{2b}$$

Stationary Vortices

For static configurations, the in-plane angle ϕ satisfies Laplace's equation: $\vec{\nabla}\cdot\vec{\nabla}\phi = 0$. A vortex centered at position $\vec{X} = (X_1, X_2)$ causes a gradient

$$\vec{\nabla}\phi(\vec{x}) = \frac{q\hat{e}_z \times (\vec{x}-\vec{X})}{|\vec{x}-\vec{X}|^2}. \tag{3}$$

The circulation charge is $q = \pm$integer, for vortices (+) or antivortices (−). While no source term appears on the RHS of the Laplace equation for ϕ, we have

$$\vec{\nabla}\times\vec{\nabla}\phi = 2\pi q\delta(\vec{x}-\vec{X})\hat{e}_z. \tag{4}$$

This can be expressed in the form of a Gauss Law by rotating through 90^o, to make what is called the stream potential Ψ in fluid mechanics,

$$\vec{\nabla}\Psi = \vec{\nabla}\phi \times \hat{e}_z, \tag{5a}$$

$$\vec{\nabla}\cdot\vec{\nabla}\Psi = 2\pi q\delta(\vec{x}-\vec{X}). \tag{5b}$$

From these it is clear that q is a conserved charge with only integer values.

The out-of-plane component S^z determines the type of vortex. When the equations (2) are <u>linearized</u> in S^z, one finds $S^z \approx \dot{\phi}/J$, which gives $S^z = 0$ in the static limit and defines the static <u>in-plane</u> vortex. If the nonlinear terms in S^z are kept, the static out-of-plane vortex with $S^z \neq 0$ results. Far from the vortex core, the out-of-plane vortex has asymptotic form[9]

$$S^z \to p\sqrt{\frac{r_v}{|\vec{x}-\vec{X}|}}\, e^{-|\vec{x}-\vec{X}|/r_v}, \tag{6a}$$

$$r_v \equiv \frac{1}{2}\sqrt{\frac{1-\delta}{\delta}}. \tag{6b}$$

A natural length scale is determined by the vortex radius r_v, and $p = \pm 1$ determines the sign of S^z at the vortex core. Simulations[12] and energy estimates[9] have shown that only the in-plane vortex is stable for strong anisotropy ($\delta > 0.28$ for square lattice), while only the out-of-plane vortex is stable for weak anisotropy ($\delta < 0.28$ for square lattice).

Moving Vortices

When either type of vortex moves, perhaps due to the effect of other vortices, the out-of-plane spin component aquires a change S_1^z proportional to the vortex velocity $\vec{V}$ for low speed. Far from the vortex, with vortex position $\vec{X}(t) = \vec{V}t$, we have[9]

$$S_1^z \approx \frac{q\vec{V}\times(\vec{x}-\vec{X}(t))}{4J\delta|\vec{x}-\vec{X}(t)|^2}\cdot\hat{e}_z. \tag{7}$$

It is these structural changes in the vortex form that account for the generation of the mass to be discussed below.

THIELE EQUATION

Thiele[7] derived an equation of motion for a domain wall acted on by an external force $\vec{F}$, that was later applied to vortices by Huber. The Thiele equation follows from the Landau-Lifshitz equation (2) with the assumption that the vortex spin field does not depend explicitly on time, but only implicitly due to the velocity via $X(t) = V(t)t$, such that $\vec{S}(\vec{x}, t) = \vec{S}(\vec{x} - \vec{V}t)$. The Thiele equation is

$$\vec{F} + \vec{G} \times \vec{V} = 0, \tag{8a}$$

$$\vec{F} = -\frac{\partial H}{\partial \vec{X}}, \tag{8b}$$

$$\vec{G} = S^{-2} \int d^2x \; \vec{S} \cdot (\partial_1 \vec{S} \times \partial_2 \vec{S}) \; \hat{e}_z = \int d^2x \; \vec{\nabla}\phi \times \vec{\nabla} S^z. \tag{8c}$$

$\vec{F}$ is the net force on the vortex due to other vortices or external fields, and the gyrovector $\vec{G}$ is a topological invariant for the system as a whole in the continuum limit. However, since the S^z component for a vortex is localized, a gyrovector is defined also for an individual vortex. For out-of-plane vortices, $\vec{G} = 2\pi pq S \hat{e}_z$, and the equation gives a good description of the dynamics.[13] To the contrary, for in-plane vortices, $\vec{G} = 0$, and the equation makes no sense, since $\vec{F}$ need not be zero.

It is interesting to consider why the equation is invalid for in-plane vortices. The difficulty comes from the fact that the S^z component is zero for the static vortex, and then changes appreciably (in a relative sense) with velocity. Then the basic assumption of a fixed vortex shape that simply translates is strongly violated. Thus it is necessary to modify the derivation of the Thiele equation to include these velocity-dependent structural changes.[10]

GENERALIZED THIELE EQUATION

In addition to implicit time dependence, it is necessary to include an explicit time dependence of the spin field. The greatest explicit time dependence comes from a velocity that changes with time, $\vec{V}(t)$. While the in-plane angle has changes that are second order in $\vec{V}$, S^z is changed to first order in $\vec{V}$. This is somewhat like the velocity dependence of the electric and magnetic fields of a moving charge. In this spirit the spin field's time dependence is assumed to be carried by the vortex position $\vec{X}(t)$ and velocity $\vec{V}(t)$,

$$\vec{S}(\vec{x}, t) = \vec{S}(\vec{x} - \vec{X}(t), \vec{V}(t)). \tag{9}$$

Following Thiele, one can contract a spatial gradient $\partial_i \vec{S} \equiv \partial \vec{S}/\partial x_i$, where $i = 1, 2$, with $\vec{S} \times \dot{\vec{S}}$, and make use of the replacement, $\partial \vec{S}/\partial x_i = -\partial \vec{S}/\partial X_i$. There results an equation in force densities,

$$\vec{S} \cdot \left(\frac{\partial \vec{S}}{\partial x_i} \times \frac{d\vec{S}}{dt} \right) = -S^2 \frac{\delta H}{\delta \vec{S}} \frac{\partial \vec{S}}{\partial x_i} = +S^2 \frac{\delta H}{\delta X_i}. \tag{10}$$

But the time derivative of the spin field is equivalent to a combination of gradients with respect to $\vec{x}$ and vortex velocity;[10]

$$\frac{d}{dt}\vec{S}(\vec{x},t) = -\frac{\partial \vec{S}}{\partial x_j}\frac{dX_j}{dt} + \frac{\partial \vec{S}}{\partial V_j}\frac{dV_j}{dt}. \tag{11}$$

Using equation (11) in (10) and integrating over area leads to the generalized Thiele equation,

$$\vec{F} + \vec{G} \times \vec{V} = \hat{e}_i M_{ij}\frac{dV_j}{dt}, \tag{12}$$

where $\vec{F}$ and $\vec{G}$ are defined in (8) and M is the effective mass tensor with elements,

$$M_{ij} = -S^{-2}\int d^2x\ \vec{S}\cdot\left(\frac{\partial \vec{S}}{\partial x_i}\times\frac{\partial \vec{S}}{\partial V_j}\right) \tag{13}$$

Although the gyrovector can be evaluated for an arbitrary spin configuration, the mass tensor depends on a derivative with respect to a collective coordinate, the vortex velocity, and can be evaluated only once a velocity-dependent vortex solution is known. For the slowly moving in-plane or out-of-plane vortex, with ϕ and S_1^z given in (3) and (7), the gradients $\partial\phi/\partial\vec{x}$ and $\partial S^z/\partial\vec{V}$ are parallel, and assuming the dominant contributions come from large radius, one finds that M is diagonal and can be replaced by a scalar;

$$M_{i,j} = M\ \delta_{ij} \approx \frac{\pi q^2}{4J\delta}\ln(L/a_o)\delta_{ij} \tag{14}$$

L is the system radius and a_o is a short distance core cutoff. The mass is found to be proportional to the vortex creation energy, and diverges in the same logarithmic sense. However, it is not clear whether contributions from near the core that have not been included could cancel this divergence.

Use of Canonical Fields ϕ, S^z

The generalized Thiele equation also can be derived very efficiently starting from the Hamilton equations for the canonical fields ϕ and S^z. The same velocity-dependent travelling wave ansatz (9) is assumed, together with equation (11) for the time derivative. Instead of the Landau-Lifshitz equation, we use the canonical equations of motion for the total time derivatives and equate to expression (11),

$$\frac{d\phi}{dt} = \frac{\delta H}{\delta S^z} = -V_j\frac{\partial\phi}{\partial x_j} + \frac{dV_j}{dt}\frac{\partial\phi}{\partial V_j}, \tag{15a}$$

$$\frac{dS^z}{dt} = -\frac{\delta H}{\delta\phi} = -V_j\frac{\partial S^z}{\partial x_j} + \frac{dV_j}{dt}\frac{\partial S^z}{\partial V_j}. \tag{15b}$$

Taking (15a) times $\partial S^z/\partial x_i$ and (15b) times $-\partial\phi/\partial x_i$ and summing the two equations gives again the generalized Thiele equation, in terms of a gyrotensor G_{ij} and mass tensor,

$$F_i + G_{ij}V_j = M_{ij}\frac{dV_j}{dt}, \tag{16a}$$

$$F_i = \int d^2x \left(\frac{\delta H}{\delta \phi} \frac{\partial \phi}{\partial x_i} + \frac{\delta H}{\delta S^z} \frac{\partial S^z}{\partial x_i} \right) = -\frac{\partial H}{\partial X_i}, \tag{16b}$$

$$G_{ij} = \int d^2x \left(\frac{\partial \phi}{\partial x_i} \frac{\partial S^z}{\partial x_j} - \frac{\partial \phi}{\partial x_j} \frac{\partial S^z}{\partial x_i} \right), \tag{16c}$$

$$M_{ij} = -\int d^2x \left(\frac{\partial \phi}{\partial x_i} \frac{\partial S^z}{\partial V_j} - \frac{\partial \phi}{\partial V_j} \frac{\partial S^z}{\partial x_i} \right). \tag{16d}$$

These expressions are equivalent to those already given. The gyrovector and this gyrotensor are related via $G_{ij} = \epsilon_{ij3} G$.

Multi-Vortex Dynamics

The above derivations can be generalized to multiple-vortex configurations, by contracting with gradients with respect to the vortex positions rather than with respect to space position $\vec{x}$. There results effective mass and gyrovector terms due to pairs of vortices as well as those due to individual vortices.[14] These additional terms also have important consequences for dynamics.[15]

Discrete Gyrovector

The gyrovector defined in (8c) can be shown to be conserved. But physical spin systems exist on a lattice. Therefore it is necessary to consider the generalization of (8c) onto a lattice, and question whether it is still conserved. The lowest order symmetrical finite difference approximation for G on a square lattice is

$$G = (2S)^{-2} \sum_{\mathbf{n}} \vec{S}_{\mathbf{n}} \cdot (\vec{S}_{\mathbf{n+a}} - \vec{S}_{\mathbf{n-a}}) \times (\vec{S}_{\mathbf{n+b}} - \vec{S}_{\mathbf{n-b}}), \tag{17}$$

where $\mathbf{a} = a\hat{e}_1, \ \mathbf{b} = b\hat{e}_2$ are the lattice basis vectors. For other lattices, there will be a similar sum over triple products of spins in all possible triangular plaquettes. Using the discrete equations of motion that follow from (1), one can show that generally $dG/dt \neq 0$. This result is seen in vortex simulations. Furthermore, when the total $G \neq 0$, it allows for vortex-antivortex annihilation in the discrete system that would be prohibited in the continuum limit.

Relation to Guiding Center

The generalized Thiele equation has the same form as that for a charge e in uniform electric and magnetic fields $\vec{E}$ and $\vec{B}$, with the identifications, $\vec{F} \to e\vec{E}, \ \vec{G} \to -e\vec{B}$. When $|\vec{E}| < |\vec{B}|$, it is possible to transform to a frame where the electric field vanishes,[16] moving at relative velocity $\vec{U} = \vec{E} \times \vec{B}/B^2$ or $\vec{U} = \vec{F} \times \vec{G}/G^2$ in vortex language. This relative velocity can be thought of as the velocity of a "guiding center" for the vortex, about which the vortex core may make cyclotron oscillations, in analogy to electrodynamics. The generalized Thiele equation can be put in guiding center form by writing it with only first order time derivatives. Define guiding center coordinate $\vec{R}$;

$$R_1 = X_1 - \frac{M}{G} V_2, \tag{18a}$$

$$R_2 = X_2 + \frac{M}{G} V_1. \tag{18b}$$

Then the dynamics for the guiding center is identical to the Thiele equation,

$$\vec{F} + \vec{G} \times \frac{d\vec{R}}{dt} = 0, \tag{19}$$

and the solution for constant force is $\vec{U} \equiv \; d\vec{R}/dt = \vec{F} \times \vec{G}/G^2$, as above.

For in-plane vortices, which have $\vec{G} = 0$, the transformation to guiding center coordinate cannot be made, since it is analogous to the case $|\vec{E}| > |\vec{B}|$ in electrodynamics. Then the equation of motion for the core position must take the form of Newton's law,

$$\vec{F} = M \frac{d\vec{V}}{dt}. \tag{20}$$

On the other hand, for the out-of-plane vortices, when the mass is small the cyclotron motion might not be seen due to discreteness effects. Then the Thiele equation without mass may be accurate except for describing the discreteness effects. However, in general, then mass should be included for both types of vortices.

VORTEX MOMENTUM

A Lagrangian for this spin model which leads to (2) is

$$L = \int d^2x \; S^z \frac{d\phi}{dt} \; - \; H. \tag{21}$$

This suggests the following definition for a momentum functional of the spin field, with $S^z \to 0$ at infinity,

$$\vec{P} = -\int d^2x \; S^z \vec{\nabla}\phi. \tag{22}$$

This type of definition has been used before in discussions of momentum for solitons in 1D magnets.[17] $\vec{P}$ can be shown directly to be a generator of space translations, by considering the Poisson bracket with an arbitrary function $f(\vec{x})$. The fundamental relation is

$$\{\phi(\vec{x}), S^z(\vec{x}\,')\} = \delta(\vec{x} - \vec{x}\,'), \tag{23}$$

so there results

$$\{\vec{P}, f(\vec{x})\} = \vec{\nabla} f(\vec{x}). \tag{24}$$

One can also determine the Poisson bracket between the two components of $\vec{P}$, which is found to be nonzero:

$$\{P_1, P_2\} = \int d^2x \; \left(\frac{\delta P_1}{\delta\phi} \frac{\delta P_2}{\delta S^z} - \frac{\delta P_1}{\delta S^z} \frac{\delta P_2}{\delta\phi} \right) = \int d^2x \; \left(\vec{\nabla}\phi \times \vec{\nabla} S^z \right) \cdot \hat{e}_z = G. \tag{25}$$

Then $\vec{P}$ cannot be a canonical momentum because this does not vanish. However, for a moving in-plane or out-of-plane vortex, to lowest order in velocity, only S_1^z and the static in-plane profile contribute to the integral, giving

$$\vec{P} = \frac{\pi q^2}{4J\delta} \ln{(L/a_o)} \, \vec{V}. \tag{26}$$

The result is identical to the lowest order approximation for the effective mass times the velocity.

This type of momentum has been used for 2D spin dynamics with easy-axis as well as easy-plane anisotropy. Ivanov and Stephanovich[18] considered the time derivative of a similar momentum expression for the easy-axis case, where we should replace S^z by $S^z \pm S$ in (22), depending on the boundary condition at infinity. Wysin and Mertens[10] have considered the time derivative for the easy-plane case. Both cases appear to generate a term dependent on the gyrovector, but one needs to be careful in performing the time derivatives. If we assume that the spin field has a translational dependence as well as some internal time-dependence, i. e. , $\vec{S}(\vec{x} - \vec{X}(t), t)$, then we can proceed to differentiate (22), where the time derivative includes a convective term,

$$\frac{d}{dt} = \frac{\partial}{\partial t} - V_j \partial_j. \tag{27}$$

For a particular component of $\vec{P}$, we have

$$\frac{dP_i}{dt} = -\int d^2x \, \left\{ \left(\frac{\partial S^z}{\partial t} - V_j \partial_j S^z \right) \partial_i \phi + S^z \left(\frac{\partial}{\partial t} \partial_i \phi - V_j \partial_j \partial_i \phi \right) \right\}. \tag{28}$$

Now if a vortex is present somewhere, then it is not possible to interchange the order of the space derivatives in its vicinity, because of the Gauss relations (4) and (5). However, we can rewrite the last term using

$$\partial_j \partial_i \phi = \partial_i \partial_j \phi - \epsilon_{ij3} (\vec{\nabla} \times \vec{\nabla} \phi) \cdot \hat{e}_z. \tag{29}$$

Then integrating by parts and re-arranging gives

$$\frac{dP_i}{dt} = \int d^2x \, \{ \left(\frac{-\partial S^z}{\partial t} \partial_i \phi + \frac{\partial \phi}{\partial t} \partial_i S^z \right)$$

$$+ V_j \left[(\partial_i \phi \partial_j S^z - \partial_i S^z \partial_j \phi) - S^z \epsilon_{ij3} (\vec{\nabla} \times \vec{\nabla} \phi) \cdot \hat{e}_z \right] \}. \tag{30}$$

Ivanov and Stephanovich identify the partial time derivatives with variations of the energy,

$$\frac{\partial \phi}{\partial t} = \frac{\delta H}{\delta S^z}, \qquad \frac{\partial S^z}{\partial t} = -\frac{\delta H}{\delta \phi}. \tag{31}$$

With use of the equations of motion, the first term in (30) reduces to a force-like term, the negative gradient of H with respect to vortex position. The second term is proportional to the gyrovector, but so is the last term, coming in with the opposite sign, and these two cancel each other out. Then we would get simply

$$\frac{dP_i}{dt} = -\frac{\partial H}{\partial X_i}. \tag{32}$$

However, this is inconsistent with $\vec{P} = M\vec{V}$ in (26) and the generalized Thiele equation (12). It seems that the energy variations in equation (31) must be identified with the *total* time derivatives of ϕ and S^z. Then the generalized Thiele equation can be derived directly using the canonical fields ϕ and S^z as shown above.

Another Momentum Functional

Papanicolaou and Tomaras[19] have also considered how to define magnetic vortex momentum, especially for systems with uniaxial anisotropy. They defined the following momentum functional

$$\vec{T} = \int d^2x \; \vec{x} \times \vec{g},$$

$$\vec{g} = \vec{\nabla}\phi \times \vec{\nabla}S^z. \tag{33}$$

$\vec{g}$ represents the gyrovector density. $\vec{g}$ has only a z component for 2d magnets, but these definitions may also be used for 3d.

For a slowly moving in-plane vortex, far from the vortex core the gyrodensity due to the motion is

$$\vec{g}_1 = \frac{q^2}{4J\delta} \frac{\vec{V} \times (\vec{x} - \vec{X})}{|\vec{x} - \vec{X}|^4}. \tag{34}$$

An out-of-plane vortex also has a similar perturbation $\vec{g}_1$ away from its static gyrodensity $\vec{g}_o$. The total gyrodensity would be $\vec{g} = \vec{g}_o + \vec{g}_1$, where $\vec{g}_o$ is isotropic around the vortex core, and $\vec{g}_1$ is antisymetric about a line along the velocity. Then for either type of moving vortex,

$$\vec{T} = \vec{X} \times \vec{G} + M\vec{V}, \tag{35}$$

where M is given in (14). This momentum depends on the choice of origin of the system, as well as the velocity. If the time derivative of $\vec{T}$ is set equal to any external force acting on the vortex, the generalized Thiele equation results: $\vec{F} + \vec{G} \times \vec{V} = M\dot{\vec{V}}$.

The two components of $\vec{T}$ have the Poisson bracket, $\{T_1, T_2\} = G$. However, $\vec{T}$ is not a generator of translations, we find the following relationship:

$$\{\vec{T}, f(\vec{x})\} = \vec{\nabla} f - \vec{x} \times \left(\frac{\delta f}{\delta \phi} \vec{\nabla} \times \vec{\nabla}\phi + \frac{\delta f}{\delta S^z} \vec{\nabla} \times \vec{\nabla} S^z \right). \tag{36}$$

If a vortex is present then the second term need not be zero, but can contribute a delta function according to Eq. (4).

CONCLUSION

The Thiele equation of motion is applicable if the vortex motion is very smooth, without structural changes in the vortex profile. But generally these structural changes occur if the vortex accelerates, and lead to generation of a mass. The presence of the mass is predicted on the basis of dynamics obtained from the Landau-Lifshitz equation of motion for the spin dynamics. The mass will be responsible for more complicated vortex motion, and in particular, for adding cyclotron-like motions for out-of-plane vortices. We expect that the mass varies as δ^{-1} as $\delta \to 0$, but this needs more careful study. The mass should also relate to a vortex momentum, but it was not necessary to define vortex momentum to obtain the generalized Thiele equation. The discretized version of the gyrovector on a lattice has been found to be nonconserved, probably due to the large spatial gradients in the spin field near the vortex core. Some ideas for vortex momentum have been presented, but a complete description of vortex momentum is not yet complete.

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