

## Collective Variable Approach to the Dynamics of Non-linear Magnetic Excitations with Application to Vortices

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### Abstract

We propose a collective variable ansatz for a system of  $N$  extended, but finite, non-linear excitations in magnetic systems. In contrast to earlier approaches, where the interactions between the different excitations have been treated only through an external force term, we explicitly consider a dependence of the microscopic spin field on all the coordinates and velocities of the localized objects. This leads to  $N$  coupled equations of motion with parameters (mass and gyro tensors) which explicitly depend on the mutual distances of the excitations. We apply this ansatz to vortices in two-dimensional Heisenberg ferromagnets with weak easy-plane anisotropy. For vortex pairs we find either rotational or translational motion, with an additional cyclotron-like oscillation on top of the main trajectories. Due to the interactions between the two vortices we obtain two different eigen-values of the mass and gyro tensors with values depending on the distance between the vortices, their vorticities, and the sign of their out-of-plane structures. In contrast to the single vortex mass, which depends logarithmically on the system size  $L$ , we find two-vortex masses, which are independent of  $L$ , but depend on their mutual distance. These predictions are in good qualitative agreement with numerical simulations of the complete spin systems. However, since these simulations are performed at zero temperature, where the vortex pairs extend throughout the whole system, we observe a strong influence of the boundaries on the

absolute values of the masses.

Majorana Neutrinos with Application to Fermions

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The following is a summary of the main results of the paper.

Abstract

The following is a summary of the main results of the paper. The paper discusses the possibility of Majorana neutrinos and their application to fermions. It covers topics such as mass generation, neutrino oscillations, and the implications for the Standard Model. The authors present theoretical arguments and calculations related to these topics.

## I. INTRODUCTION

Non-linear excitations play an important role in many different areas of physics, which has led to an increasing interest in these objects in the last 30 years<sup>1,2</sup>. Unfortunately, only a few non-linear systems – most of them one-dimensional (1d) – can be treated exactly. In general one has to develop approximate methods to understand the dynamics of these systems. For the case of well-localized non-linear excitations *collective variable* approaches have been proven quite successful for many applications<sup>3</sup>. In these approaches, the non-linear excitations are treated as particle-like structures (on a mesoscopic length scale) with well-defined positions and velocities. Even their interaction with the surrounding fluctuations can be incorporated to some degree<sup>1,3</sup>.

Magnetic systems are particularly useful for studying non-linear phenomena, because (i) they allow for many different non-linear excitations, e.g. domain walls, vortices, bubbles, etc., depending on the form of the mutual interaction of the magnetic moments and with applied external fields; (ii) they can be mapped, at least in the 1d case, for some limiting cases onto completely integrable systems; (iii) they are relatively easy to simulate on computers, even for physically relevant system sizes, i.e., larger than the characteristic length scales of the quantities under investigation.

We will focus here on 2d magnetic systems which can show, due to their intermediate dimensionality, a quite genuine dynamical behavior<sup>6</sup>. For example, Heisenberg magnets with XY- or easy-plane symmetry (i.e. the interaction of the  $z$ -components of the spins is smaller than the one between the other components) exhibit a “topological” phase transition, related to the unbinding of vortex-antivortex pairs above a critical temperature  $T_{KT}$ , which can be observed only in 2d systems<sup>7-9</sup>. Vortices are topological, non-linear excitations characterized by a charge  $q$ . The impact of vortex-antivortex pairs below  $T_{KT}$  on the spin dynamics, as measured e.g. by dynamical correlation functions, leads only to a renormalization of the spin wave peak<sup>10</sup>. Above the transition temperature, however, the unbound moving vortices contribute to an additional central peak which has been observed both in

computer simulations<sup>11</sup> and neutron scattering experiments on quasi-2d materials<sup>12</sup>. Analytic calculations by Mertens et al.<sup>13</sup> predicted a quadratic Lorentzian shape of this central peak. In this approach the free vortices were considered to move ballistically with a Boltzmann velocity distribution characterized by an average thermal speed  $v_{rms}$ . The thermal speed was estimated by Huber<sup>14</sup> from a velocity auto-correlation function based on a vortex equation of motion which was derived by Thiele<sup>15</sup> using a collective variable approach for localized structures in magnets. This calculations of  $v_{rms}$ , however, did not take into account the presence of different vortex structures for different strengths of the easy-plane anisotropy. Moreover, Thiele's approach is for a single, well-localized excitation only, while its interactions with other excitations are treated through an external force term.

Vortices, however, have an infinitely extended static in-plane structure which also results in an extended perturbation of the static out-of-plane structure in the case of finite velocity. Though the superposition of the static in-plane structure of two vortices can be represented as a central force between their centers<sup>7</sup>, this is no longer true for the superposition of their velocity-induced out-of-plane structures. These corrections to the static vortex structure show up in the calculation of the mass and the gyro tensor of the Thiele equation and suggest that the single-vortex approach is no longer sufficient for an understanding of many-vortex dynamics. For finite temperatures the vortex size becomes finite due to screening<sup>16</sup> (i.e. its structure is now localized), and a collective variable approach seems to be appropriate to construct equations of motions for the vortex dynamics. However, as long as vortex radii are not much smaller than their average separations we still have to include the effects of the overlapping fields in understanding the many-vortex dynamics.

We introduce here a collective variable approach for  $N$  non-linear excitations in magnetic systems. We derive  $N$  coupled equations of motion, each one of them similar to the original Thiele equation, but with additional inertial terms, resulting from an explicit dependence of the microscopic spin field on the vortex velocities. We then consider explicitly vortices in 2d easy-plane ferromagnets, which, for small anisotropies, already have statically a well-pronounced out-of-plane structure. Hence, each of them is characterized by two independent

“charges”, namely its vorticity  $q$  and the sign of its static out-of-plane component  $p$ . We use approximate solutions<sup>17</sup> to calculate the mass and gyro tensors of this system, which now become functions of the mutual vortex distance. The discussion of the dynamics becomes simplest for two vortices: they either rotate around each other ( $q^1 p^1 = q^2 p^2$ ), or they move parallel to each other ( $q^1 p^1 = -q^2 p^2$ ), with an additional small cyclotron-like oscillation on top of these trajectories. Due to the two-vortex corrections of the mass and gyro tensors this dynamics is now determined by two different masses which influence the amplitudes and frequencies differently for the various cases. This is in contrast to the one-vortex calculations, where one expects only one vortex mass. Comparison with numerical two-vortex simulations show good qualitative agreement with most of our analytic predictions. However, because we necessarily performed the simulations on a finite lattice at zero temperature, we also find an influence of the boundaries on the vortex parameters, which in some cases, is quite strong.

The paper is organized as follows: In Section 2 we will briefly review the original approach of Thiele which led to an equation of motion for a single vortex. We also include a short discussion of a generalization of this ansatz including inertial effects which finally lead to a vortex mass. In Section 3 we generalize the Thiele ansatz to a set of  $N$  localized magnetic structures. An application of these results to vortices in easy-plane Heisenberg ferromagnets is demonstrated in Section 4. In Section 5 we discuss the special case of just two vortices, and briefly compare our analytic results with numerical simulations. Section 6 contains a short summary.

## II. SINGLE-VORTEX EQUATION OF MOTION

We will consider here magnetic systems which are described by the Landau-Lifshitz equation (LLE) with an additional Gilbert damping term

$$\dot{\mathbf{S}} = \mathbf{S} \times \left( -\frac{\delta \mathcal{H}}{\delta \mathbf{S}} \right) - \alpha \mathbf{S} \times \dot{\mathbf{S}}; \quad (1)$$

$\mathbf{S}$  is a classical magnetization vector in units of  $\hbar$ .  $\mathcal{H}$  is the energy of the system, and  $\alpha$  is a measure of the strength of the damping term. Eqn. (1) preserves the length of the spins, hence we can parametrize the vector  $\mathbf{S}(\mathbf{r})$  with two canonically conjugated fields, the in-plane angle  $\phi(\mathbf{r})$  and the  $z$ -component of the spin field  $m(\mathbf{r}) \equiv S_z(\mathbf{r}) = S \sin \theta(\mathbf{r})$  with the out-of-plane angle  $\theta(\mathbf{r})$ :

$$\mathbf{S} = (\sqrt{S^2 - m^2} \cos \phi, \sqrt{S^2 - m^2} \sin \phi, m). \quad (2)$$

Starting from Eqn. (1) Thiele<sup>15</sup> has derived an equation of motion for a single, well-localized excitation with center position  $\mathbf{X}(t)$  under the assumption of steady-state motion (i.e.  $\mathbf{S}(\mathbf{r}, t) = \mathbf{S}(\mathbf{r} - \mathbf{X}(t))$  and  $\dot{\mathbf{X}} = \text{const.}$ ); we refer to this as the Thiele-equation (TE)

$$\mathbf{F} + \mathbf{g} \times \dot{\mathbf{X}} - \mathbf{D}\dot{\mathbf{X}} = 0, \quad (3)$$

where, for a 2d system,

$$\mathbf{F} = - \int d^2r \nabla \dot{\mathcal{H}}, \quad (4a)$$

$$\mathbf{g} = \int d^2r \nabla \phi \times \nabla m, \quad (4b)$$

$$\mathbf{D} = \alpha \int d^2r \left\{ \frac{\nabla m \nabla m}{S^2 - m^2} + (S^2 - m^2) \nabla \phi \nabla \phi \right\}. \quad (4c)$$

$\mathbf{F}$  is the net force on an excitation due to other excitations or external fields,  $\dot{\mathcal{H}}$  is the energy density of the system, the gyrovector  $\mathbf{g}$  is a topological invariant for the system as a whole,  $\mathbf{D}$  is the damping matrix, and  $\nabla$  is the gradient with respect to  $\mathbf{r}$ .

This approach was applied to single vortices in 2d easy-plane Heisenberg ferromagnets<sup>14,18</sup> (HFM) with Hamiltonian

$$\mathcal{H} = \frac{J}{2} \int d^2r \left\{ (\nabla \mathbf{S})^2 + 4\delta S_z^2 - \delta (\nabla S_z)^2 \right\} \quad (5)$$

(which is the continuum limit of the discrete model  $\mathcal{H} = \sum_{\langle ij \rangle} (\mathbf{S}_i \mathbf{S}_j - \delta (S_z)_i (S_z)_j)$ ).  $0 < \delta \leq 1$  measures the anisotropy of the system with the two limiting cases  $\delta = 0$  and  $\delta = 1$

corresponding to the isotropic and planar Heisenberg models, respectively. Depending on  $\delta$  there exist two types of stable vortex structures with distinct dynamics<sup>17,19</sup>. For  $\delta \gtrsim \delta_c$  ( $\approx 0.29$  on a square lattice) static vortices are purely in-plane; for  $\delta < \delta_c$  an out-of-plane component develops. The size of this  $S^z$  component increases with decreasing  $\delta$ , allowing for a continuous crossover to the isotropic Heisenberg limit, where the topological excitations are instantons and merons rather than vortices. A single static vortex at position  $\mathbf{X}(t) = (X_1, X_2)$  (lower indices denote the vector components) has the form<sup>17</sup>

$$o_s(\mathbf{r} - \mathbf{X}) = q \arctan \frac{y - X_2}{x - X_1}; \quad m_s(\mathbf{r} - \mathbf{X}) = pm_0(|\mathbf{r} - \mathbf{X}|), \quad (6)$$

where  $m_0(r)$  has a well-localized peak at  $r = 0$ , while its large- $r$  limit is given by

$$m_0(r) \approx \sqrt{\frac{r_v}{r}} e^{-r/r_v} \quad (7)$$

with the vortex core radius  $r_v = \frac{1}{2} \sqrt{\frac{\lambda}{1-\lambda}}$  (Fig. 1a). For  $\delta < \delta_c$  the so-called polarization  $p = \pm 1$  denotes the sign of the static out-of-plane structure, while  $p = 0$  for  $\delta > \delta_c$ . This static structure becomes distorted when the vortex is moving and in lowest order in velocity this change in its shape is<sup>17</sup>

$$o_v(\mathbf{r} - \mathbf{X}; \dot{\mathbf{X}}) = p \hat{o}(|\mathbf{r} - \mathbf{X}|) \frac{(\mathbf{r} - \mathbf{X}) \cdot \dot{\mathbf{X}}}{|\mathbf{r} - \mathbf{X}|}, \quad \hat{o}(r) = Km_0(r), \quad (8a)$$

$$m_v(\mathbf{r} - \mathbf{X}; \dot{\mathbf{X}}) = qC \hat{m}(r - \mathbf{X}) \frac{[(\mathbf{r} - \mathbf{X}) \times \dot{\mathbf{X}}] \cdot \mathbf{e}_z}{|\mathbf{r} - \mathbf{X}|^2}, \quad (8b)$$

with  $K = \frac{r_v}{J_S}$  and  $C = (4\delta JS^2)^{-1}$ . The velocity-induced change in the in-plane field is as equally localized as the static out-of-plane structure. The change in  $m(\mathbf{r})$ , on the other hand, is decaying only as  $\frac{1}{r}$  (i.e.  $\hat{m}(r \gg 1) \cong 1$ ). Close to the core its structure is shown in Fig. 1b. Since the main part of the angular anisotropy is evidently to be covered by the cross product  $(\mathbf{r} - \mathbf{X}) \times \dot{\mathbf{X}}$ , we will simplify our following calculations by assuming  $\hat{m}$  to be a function of the radial coordinate  $r$  alone. This angular dependence of  $m_v(\mathbf{r})$  is not only observed for a single moving vortex. Simulations of vortex pairs exhibit an angular dependence of the spin field which, in first and dominant order, can be well described by

a linear superposition of single vortex contributions. Hence, we are confident that a use of this symmetry in the two-vortex integrals in Sect. 4 is appropriate.

Inserting (6) and (8) into (4) (with  $\alpha = 0$ ) yields an equation of motion which can quite successfully describe the vortex dynamics with (sufficiently strong) damping, as was shown explicitly through numerical simulations of the LLE for two vortices<sup>19</sup>. However, if we set the damping parameter  $\alpha = 0$ , the TE no longer makes sense for in-plane vortices ( $\delta > \delta_c$ ), because here the gyro vector vanishes and  $\mathbf{F}$  need not be zero. It is interesting to consider why the TE is invalid for in-plane vortices. The difficulty stems from the fact that the  $z$ -component is zero for the static vortex, and then changes appreciably (in a relative sense) with velocity. Thus, the basic assumption of a fixed vortex shape that simply makes a rigid translation is strongly violated. This problem is smaller for the out-of-plane vortices because the velocity dependent changes in  $m$  are small compared to the static out-of-plane structure. Nevertheless, the TE cannot explain why out-of-plane vortices perform small oscillations around their mean trajectories as observed in computer simulations with small or zero damping (see below).

Wysin et al.<sup>20</sup> therefore proposed a more general ansatz for the spin field where the time dependence enters explicitly not only through the vortex position, but also through its velocity:

$$\mathbf{S} = \mathbf{S}(\mathbf{r} - \mathbf{X}(t), \dot{\mathbf{X}}(t)). \quad (9)$$

Inserting (9) into (1) and following the procedure of Thiele leads to the generalized Thiele equation (GTE)

$$\mathbf{F} + \mathbf{g} \times \dot{\mathbf{X}} = \mathbf{M} \ddot{\mathbf{X}}, \quad (10)$$

where  $\mathbf{F}$  and  $\mathbf{g}$  are defined in (4), while  $\mathbf{M}$  is an effective mass tensor with components

$$\mathbf{M}_{ij} = - \int d^2r \left( \frac{\partial \phi}{\partial x_i} \frac{\partial m}{\partial \dot{X}_j} - \frac{\partial \phi}{\partial \dot{X}_j} \frac{\partial m}{\partial x_i} \right). \quad (11)$$

Because of the dependence of the fields on  $\mathbf{r} - \mathbf{X}$  we can rewrite the gradients in  $\mathbf{M}$ ,  $\mathbf{D}$ , and  $\mathbf{g}$  into gradients with respect to  $\mathbf{X}$  ( $\frac{\partial}{\partial \mathbf{r}} \rightarrow -\frac{\partial}{\partial \mathbf{X}}$ ). This form of the mass tensor and



gyro vector (i.e. a pure dependence on gradients of the fields with respect to the collective variable  $\mathbf{X}$ ) can also be obtained by using the more generalized ansatz  $\mathbf{S} = \mathbf{S}(\mathbf{r}; \mathbf{X}, \dot{\mathbf{X}})$ , if one immediately uses the gradient with respect to  $\mathbf{X}$  in the Thiele ansatz. This fact will allow us to generalize Thiele's ansatz to more than one excitation (see Sec. 3).

For a single vortex system the mass tensor is diagonal:  $M_{ij} = M_0 \delta_{ij}$  with  $M_0$  given in Eqn. (18a). Since we do not know the exact form of the field  $m(r)$ , we can try to determine the mass through numerical simulations of the complete spin system (the gyro vector can be easily evaluated analytically, giving  $\mathbf{g} = 2\pi q p \mathbf{e}_z$ ). Without an external field the force  $\mathbf{F}$  is zero and the vortex is expected to perform a circular motion with angular frequency  $\omega_c = \frac{|\mathbf{g}|}{M_0}$ . However, this is difficult to test by direct numerical simulations: Because the LLE (1) is of first order in time we need the exact shape of a moving vortex as an initial condition. But this shape is known only in the continuum limit and only far away from the vortex center. The situation is more favourable for a two-vortex system, where the two vortices drive each other, leading to a steady-state motion after a short transient time of adaption to the lattice. Because of the extended structure of the vortices their mutual interaction is not only described by a central force<sup>7</sup>, but also by a renormalized mass tensor and gyro vector. A generalization of the single-structure collective variable ansatz to many excitations will be performed in the following Section.

### III. GENERALIZATION TO N LOCALIZED EXCITATIONS

The equation of motion derived in the previous Section describes only the dynamics of a single localized structure, though the influence of other similar excitations can be included in the external force term. With this strategy, however, one will neglect important informations about the parameters given as integrals over the spin field (i.e. the gyro and mass tensor). Especially, one would expect the appearance of two-center integrals representing the mutual influence of different excitations on each other. To include all these effects for a system with  $N$  well-localized excitations we will in this Section generalize the modified Thiele ansatz (9)

further, supposing an explicit dependence of the spin field on all the positions  $\mathbf{X}^\alpha(t)$  and velocities  $\dot{\mathbf{X}}^\alpha(t)$ ,  $\alpha = 1, \dots, N$ :

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{S}(\mathbf{r}; \mathbf{X}^1, \dots, \mathbf{X}^N; \dot{\mathbf{X}}^1, \dots, \dot{\mathbf{X}}^N) \quad (12)$$

(the upper indices always denote the different excitations). To derive equations of motion for the collective variables  $\mathbf{X}^\alpha$  we will make use of the canonical equations of motion for the total time derivatives of the fields  $\phi$  and  $m$ :

$$\frac{d\phi}{dt} = \sum_{j,\beta} \dot{X}_j^\beta \frac{\partial \phi}{\partial X_j^\beta} + \sum_{j,\beta} \ddot{X}_j^\beta \frac{\partial \phi}{\partial \dot{X}_j^\beta} = \frac{\delta \dot{\mathcal{H}}}{\delta m} \quad (13a)$$

$$\frac{dm}{dt} = \sum_{j,\beta} \dot{X}_j^\beta \frac{\partial m}{\partial X_j^\beta} + \sum_{j,\beta} \ddot{X}_j^\beta \frac{\partial m}{\partial \dot{X}_j^\beta} = -\frac{\delta \dot{\mathcal{H}}}{\delta \phi}. \quad (13b)$$

Multiplying (13a) by  $\frac{\delta m}{\delta X_i^\alpha}$  and (13b) by  $-\frac{\delta \phi}{\delta X_i^\alpha}$ , adding the two resulting equations, and integrating over  $\mathbf{r}$  leads to  $N$  coupled equations of motion for the  $N$  collective variables  $\mathbf{X}^\alpha(t)$ :

$$\sum_{\beta=1}^N \left\{ \mathbf{M}^{\alpha\beta} \ddot{\mathbf{X}}^\beta + \mathbf{G}^{\alpha\beta} \dot{\mathbf{X}}^\beta \right\} = -\nabla^\alpha E, \quad \alpha = 1, \dots, N, \quad (14)$$

which are all of GTE-type. The mass ( $\mathbf{M}^{\alpha\beta}$ ) and gyro tensors ( $\mathbf{G}^{\alpha\beta}$ ) are defined as

$$\mathbf{M}_{ij}^{\alpha\beta} = \int d^2r \left\{ \frac{\partial \phi}{\partial X_i^\alpha} \frac{\partial m}{\partial X_j^\beta} - \frac{\partial \phi}{\partial X_j^\beta} \frac{\partial m}{\partial X_i^\alpha} \right\}, \quad (15a)$$

$$\mathbf{G}_{ij}^{\alpha\beta} = \int d^2r \left\{ \frac{\partial \phi}{\partial X_i^\alpha} \frac{\partial m}{\partial X_j^\beta} - \frac{\partial \phi}{\partial X_j^\beta} \frac{\partial m}{\partial X_i^\alpha} \right\}, \quad (15b)$$

i.e. there exist, as expected, single and two-excitation contributions.  $\nabla^\alpha$  is the gradient with respect to  $\mathbf{X}^\alpha$  and  $E$  is the energy of the system.

#### IV. APPLICATION TO AN N-VORTEX SYSTEM

In the following we will consider explicitly a classical Heisenberg FM with weak easy-plane anisotropy ( $\delta \lesssim 0.29$ ), where the localized excitations are out-of-plane vortices as described

in Eqn. (6). This case is particularly interesting, not only because here the gyro tensors are non-zero, but also because discreteness effects are very small<sup>19</sup>, so that we can directly compare our analytical calculations to numerical simulations without the need of adding a (phenomenological) pinning potential to the equations of motion (cf. Section 5).

To allow for an analytic calculation of the mass and gyro tensors we will restrict ourselves to systems where the vortices are well separated, i.e. where their mutual distances are larger than twice the radius  $r_v$  of the static out-of-plane structures. In this case it is justified to construct the spin field as a linear superposition of single vortex solutions (6), since the static in-plane field is a solution of the Laplace equation, while the other fields are either well localized with vanishing contributions at the centers of neighboring vortices, or they are very small beyond a radius  $r_v$ .

The energy of such a  $N$ -vortex system is given by<sup>7</sup>

$$E = - \sum_{\alpha > \beta} k q^\alpha q^\beta \ln |\mathbf{X}^\alpha - \mathbf{X}^\beta| + E_c, \quad k = 2\pi JS^2, \quad (16)$$

from which we derive

$$\nabla^\alpha E = -k q^\alpha \sum_{\beta \neq \alpha} q^\beta \frac{\mathbf{X}^\alpha - \mathbf{X}^\beta}{|\mathbf{X}^\alpha - \mathbf{X}^\beta|^2}. \quad (17)$$

$E_c$  contains the contributions from the static out-of-plane vortex structures and the velocity-dependent corrections of the vortex shapes to the total energy and can be treated as a constant within our approximations.

For the calculation of the leading contribution to the gyro tensors it is sufficient to consider only the static parts of the  $\phi$  and  $S_z$ -fields, while for the mass tensors we only need the static part of the  $\phi$  and the velocity-induced part of the  $S_z$ -field (the velocity-induced part of the  $\phi$ -field is already exponentially small). If we assume that  $\hat{m}(r)$  depends on the radial coordinate  $r$  only (which seems to be justified by comparing with the numerically obtained  $\hat{m}$ -field for  $\delta = 0$  in Fig. 1b), then we can perform all of the angle integrations exactly. These tell us which of the matrix elements are zero and which are not.

For the one-vortex mass and gyro tensor we obtain the same result as Huber<sup>14</sup> and

Pokrovsky<sup>18</sup>, however, with an additional contribution from the dynamically distorted in-plane field, namely

$$\mathbf{M}_{ij}^{\alpha\alpha} = \mathbf{M}_0 \delta_{ij}; \quad \mathbf{M}_0 = \pi C \int_{a_0}^L dr \frac{\dot{m}(r)}{r} + \pi K \int_{a_0}^L dr r m_0(r) m_0'(r) \quad (18a)$$

$$\mathbf{G}_{ij}^{\alpha\alpha} = 2\pi q^\alpha p^\alpha \epsilon_{ij}; \quad (18b)$$

$\delta_{ij}$  and  $\epsilon_{ij}$  are the 2d unit and completely antisymmetric tensors, respectively,  $a_0$  is a lower cut-off of the order of a lattice constant, and  $L$  is the system size. (In the analytic calculations  $L$  is the radius of a circular symmetric plane, hence, in our numerical simulations on a square lattice we choose  $L$  to be half of the linear dimension of the system.)

The two-vortex contributions, however, depend on the choice of the coordinate system and are only diagonal for a system with axes parallel and perpendicular to the line connecting the two vortices under consideration. (In general these coordinate systems clearly do not coincide for different pairs of the same system). In the coordinate system with the first axis parallel to  $\mathbf{X}^\alpha - \mathbf{X}^\beta$ , the second perpendicular to it (which we will refer to as the main frame system of the two vortices under consideration), we find for the mass tensors

$$\mathbf{M}_{ij}^{\alpha\beta} = q^\alpha q^\beta \begin{pmatrix} \mathbf{M}_0 - \chi(d^{\alpha\beta}) + \iota(d^{\alpha\beta}) & 0 \\ 0 & \mathbf{M}_0 - \chi(d^{\alpha\beta}) - \iota(d^{\alpha\beta}) \end{pmatrix}, \quad \alpha \neq \beta, \quad (19)$$

with the abbreviations  $d^{\alpha\beta} = |\mathbf{X}^\alpha - \mathbf{X}^\beta|$ ,

$$\iota(d^{\alpha\beta}) = \frac{\pi C}{(d^{\alpha\beta})^2} \int_{a_0}^L dr r \dot{m}(r), \quad (20)$$

and

$$\chi(d^{\alpha\beta}) = \pi C \int_{a_0}^L dr \frac{\dot{m}(r)}{r} + \pi K \int_{a_0}^L dr r m_0(r) m_0'(r). \quad (21)$$

With the help of the operator  $\hat{\mathbf{P}}_{ij}^{\alpha\beta}$ , which projects out the component parallel to  $\mathbf{X}^\alpha - \mathbf{X}^\beta$  when applied to an arbitrary vector, we can rewrite (19) in the coordinate independent form

$$\mathbf{M}_{ij}^{\alpha\beta} = q^\alpha q^\beta \left\{ (\mathbf{M}_0 - \iota(d^{\alpha\beta}) + \chi(d^{\alpha\beta})) (2\hat{\mathbf{P}}_{ij}^{\alpha\beta} - \delta_{ij}) \right\}. \quad (22)$$

All the mass tensors depend on  $M_0$  which increases logarithmically with  $L$ . if we insert the asymptotic single vortex solution for a slowly moving vortex (Eqns. (6) and (8)).

For the two-vortex gyro tensors it is convenient to discuss the cases  $q^\alpha p^\beta = q^\beta p^\alpha$  (i.e.  $\mathbf{G}_{ij}^{\alpha\alpha} = \mathbf{G}_{ij}^{\beta\beta}$ ) and  $q^\alpha p^\beta = -q^\beta p^\alpha$  (i.e.  $\mathbf{G}_{ij}^{\alpha\alpha} = -\mathbf{G}_{ij}^{\beta\beta}$ ) separately. In the first case the gyro tensors are independent on the underlying coordinate system and we obtain the completely antisymmetric matrix

$$\mathbf{G}_{ij}^{\alpha\beta} = 2\pi q^\alpha p^\beta (m_0(d^{\alpha\beta}) - m_0(L))\epsilon_{ij}, \quad \alpha \neq \beta. \quad (23)$$

Since we consider only mutual vortex distances  $d^{\alpha\beta} > r_v$ , these contributions are exponentially small (cf. Eqn. (7)). For  $q^\alpha p^\beta = -q^\beta p^\alpha$  we obtain for the gyro tensors in the main frame system (cf. the discussion of the mass tensor)

$$\mathbf{G}_{ij}^{\alpha\beta} = 2\pi q^\alpha p^\beta \zeta(d^{\alpha\beta}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta(d^{\alpha\beta}) = -\frac{1}{(d^{\alpha\beta})^2} \int_{a_0}^L dr r^2 m'_0(r), \quad (24)$$

or in general

$$\mathbf{G}_{ij}^{\alpha\beta} = 2\pi q^\alpha p^\beta \zeta(d^{\alpha\beta}) \epsilon_{ik} (2\hat{\mathbf{S}}_{kj}^{\alpha\beta} - \delta_{kj}), \quad \alpha \neq \beta, \quad (25)$$

where the operator  $\hat{\mathbf{S}}_{kj}^{\alpha\beta}$  projects out the component perpendicular to  $\mathbf{X}^\alpha - \mathbf{X}^\beta$  when applied to an arbitrary vector. In this case the components of  $\mathbf{G}^{\alpha\beta}$  decay as  $(d^{\alpha\beta})^{-2}$ .

## V. COMPARISON WITH NUMERICAL SIMULATIONS OF TWO-VORTEX SYSTEMS

We will now apply our ansatz to two-vortex systems and compare the analytical results to numerical simulations of the complete spin system.

Because we have two "quantum numbers" per vortex, namely its vorticity  $q$  and the sign of its out-of-plane structure  $p$ , we have to distinguish between four physically different cases, when considering two pairs. This is different from the vortex-pair dynamics in other systems, like e.g. incompressible fluids, where each vortex is characterized by a single "quantum number", leading to only two different scenarios in a two-vortex system.

### A. The equations of motion

For a two-vortex system it is more convenient to rewrite the equations of motion (14) in center-of-mass (cms)  $\mathbf{C} = \frac{1}{2}(\mathbf{X}^1 + \mathbf{X}^2)$  and relative coordinate  $\mathbf{Y} = \mathbf{X}^1 - \mathbf{X}^2$

$$\bar{\mathbf{M}}\ddot{\mathbf{C}} + \tilde{\mathbf{M}}\ddot{\mathbf{Y}} + \bar{\mathbf{G}}^1\dot{\mathbf{C}} + \tilde{\mathbf{G}}^1\dot{\mathbf{Y}} = \frac{kq^1q^2}{Y^2}\mathbf{Y} \quad (26a)$$

$$\bar{\mathbf{M}}\ddot{\mathbf{C}} - \tilde{\mathbf{M}}\ddot{\mathbf{Y}} + \bar{\mathbf{G}}^2u\dot{\mathbf{C}} - \tilde{\mathbf{G}}^2\dot{\mathbf{Y}} = -\frac{kq^1q^2}{Y^2}\mathbf{Y}. \quad (26b)$$

(For a derivation of the above equations and the definition of the consequent mass and gyro tensors see the Appendix). In a system with the first axis parallel, the second perpendicular, to  $\mathbf{Y}$ , the mass tensors become diagonal

$$\bar{\mathbf{M}} = \begin{pmatrix} \tilde{M}_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad \tilde{\mathbf{M}} = \frac{1}{2} \begin{pmatrix} \tilde{M}_1 & 0 \\ 0 & \tilde{M}_2 \end{pmatrix}. \quad (27)$$

with

$$\tilde{M}_1 = (1 + q^1q^2)\mathbf{M}_0 - q^1q^2(\lambda - \nu) \quad (28a)$$

$$\tilde{M}_2 = (1 + q^1q^2)\mathbf{M}_0 - q^1q^2(\lambda + \nu) \quad (28b)$$

$$\hat{M}_1 = (1 - q^1q^2)\mathbf{M}_0 + q^1q^2(\lambda - \nu) \quad (28c)$$

$$\hat{M}_2 = (1 - q^1q^2)\mathbf{M}_0 + q^1q^2(\lambda + \nu). \quad (28d)$$

These are now no longer single-vortex masses, but  $\tilde{\mathbf{M}}$  can be considered as total mass of the two-vortex system,  $\tilde{M}_1$  as the reduced mass.

The dependence of the gyro tensors on the products  $q^\alpha p^\beta$  suggests that we distinguish the cases  $q^1p^2 = q^2p^1$  and  $q^1p^2 = -q^2p^1$ .

a)  $q^1p^2 = q^2p^1$ : (vortex rotation)

For this case we find for the gyro tensors

