

COLLECTIVE VARIABLE APPROACH FOR A MAGNETIC N-VORTEX SYSTEM

A.R. Völkel¹, F.G. Mertens¹, G.M. Wysin², A.R. Bishop³, H.J. Schnitzer¹

¹Physics Institute, University of Bayreuth, 95440 Bayreuth, Germany

²Physics Department, Kansas State University, Manhattan, KS 66506, USA

³Los Alamos National Laboratory, Los Alamos, NM 87545, USA

1 INTRODUCTION

Nonlinear excitations are responsible for many important dynamical properties of materials (see e.g. Bishop et al.¹). However, the treatment of the complete dynamics of appropriate models is often not possible. For the case of well-localized excitations there have been established methods which reduce all the possible degrees of freedom to a few, chosen to be sufficient to capture the intermediate ("mesoscopic") length and time scales controlled by the nonlinear coherent structures and their interactions (collective variable approach, see e.g. Kosevich²).

For magnetic systems Thiele³ suggested such an approach starting from the microscopic Landau-Lifshitz equation with Gilbert damping. Considering steady-state motion of a single localized excitation in an external field, he derived for one collective coordinate an equation of motion (hereafter, the "Thiele equation" (TE)) which contains a damping, a gyro, and an external force term. This method was applied to easy-plane Heisenberg magnets⁴, where the localized excitations are vortices, and it could be shown that the results agree with observations of computer simulations⁵. However, this approach fails, if the damping parameter is set to zero. By generalizing Thiele's approach to non-stationary motion, Wysin et al.⁶ derived a similar equation ("Generalized Thiele equation" (GTE)), but with an additional term of the form "mass times acceleration" which takes into account the inertia of the magnetization field due to changes in the velocity of the excitation.

All of these approaches were applied to single excitations only. Though the interaction with other similar excitations can be studied by treating them as external forces, we cannot expect to find all the possible collective dynamics of such a multiple-excitation system within this ansatz. Moreover, for the vortices of the easy-plane Heisenberg ferromagnet one obtains a mass which depends logarithmically on the system size and which therefore allows no dynamics at all for an infinite system.

We present here a generalization of the ansatz of Wysin et al.⁶ to a system with N distinct excitations (Sec. 2). Considering N collective coordinates we derive a system of N coupled equations of motion, all of GTE-type. In Section 3 we consider vortices in easy-plane Heisenberg ferromagnets and calculate the mass and gyro tensors. In addition to the single-vortex contributions we now also find coupling terms depending explicitly on the mutual distance and position of the vortices. Moreover, as we show for a two-vortex system (Sec. 4), there exist degrees of freedom for which the associated masses are independent of the system size. A brief comparison with numerical simulations of two-vortex systems shows that we can successfully describe the observed dynamics.

2 DERIVATION OF EQUATIONS OF MOTION

In this section we will consider a magnetic system with Hamiltonian \mathcal{H} whose dynamics is given by the Landau-Lifshitz equation

$$\dot{\mathbf{S}} = \mathbf{S} \times \left(-\frac{\delta \mathcal{H}}{\delta \mathbf{S}} \right); \quad (1)$$

\mathbf{S} is the vector of magnetization (in units of \hbar). We furthermore suppose that the system under consideration exhibits well-localized excitations which, in a first approximation, can be uniquely characterized by (collective) coordinates $\mathbf{X}^\alpha(t)$, $\alpha = 1, 2, \dots$. One can expect that for this case the field \mathbf{S} depends explicitly on those coordinates. Therefore, assuming a system of N of these structures, we propose the following ansatz

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{S}(\mathbf{r}; \mathbf{X}^1, \dots, \mathbf{X}^N; \dot{\mathbf{X}}^1, \dots, \dot{\mathbf{X}}^N), \quad (2)$$

where we have included an explicit dependence on the velocities of the excitations⁶. Generalizing the ansatz of Thiele for our choice of \mathbf{S} we can rewrite (1) in terms of force densities

$$\epsilon_{lmn} S_l \frac{\partial S_m}{\partial X_i^\alpha} \dot{S}_n = -\frac{\delta \mathcal{H}}{\delta X_i^\alpha}, \quad (3)$$

with

$$\dot{S}_n = \sum_{\beta=1}^N \sum_{j=1}^3 \left\{ \frac{\partial S_n}{\partial X_j^\beta} \dot{X}_j^\beta + \frac{\partial S_n}{\partial \dot{X}_j^\beta} \ddot{X}_j^\beta \right\}. \quad (4)$$

(Lower indices always label vector components, upper ones the different excitations). At this point it becomes clear that we must assume an explicit dependence of \mathbf{S} on the $\dot{\mathbf{X}}^\alpha$ to derive from the first order differential equation (1) the second order differential equations (3). These terms proportional to the second time derivatives of \mathbf{S} are crucial for a correct description of the dynamics for a system without damping (to which we restrict ourselves here), as will be shown below. Since Eqn. (1) preserves the length of \mathbf{S} , we can rewrite this three-dimensional vector in the two canonically conjugate fields ϕ and m :

$$\mathbf{S} = (\sqrt{S^2 - m^2} \cos \phi, \sqrt{S^2 - m^2} \sin \phi, m). \quad (5)$$

After integrating out the field parameter \mathbf{r} , we finally arrive at N equations of motion for the N collective coordinates \mathbf{X}^α :

$$\sum_{\beta=1}^N \left\{ \mathbf{M}^{\alpha\beta} \ddot{\mathbf{X}}^\beta + \mathbf{G}^{\alpha\beta} \dot{\mathbf{X}}^\beta \right\} = -\nabla^\alpha E, \quad \alpha = 1, \dots, N. \quad (6)$$

These equations have a similar form (GTE-like) as the one for a single vortex⁶, but contain additional coupling terms which can become quite important, as will be seen below. The tensors $\mathbf{M}^{\alpha\beta}$ and $\mathbf{G}^{\alpha\beta}$, which we will call mass and gyro tensors, respectively, are defined as

$$\mathbf{M}_{ij}^{\alpha\beta} = \int dV \left\{ \frac{\partial\phi}{\partial X_i^\alpha} \frac{\partial m}{\partial \dot{X}_j^\beta} - \frac{\partial\phi}{\partial \dot{X}_j^\beta} \frac{\partial m}{\partial X_i^\alpha} \right\}, \quad (7)$$

$$\mathbf{G}_{ij}^{\alpha\beta} = \int dV \left\{ \frac{\partial\phi}{\partial X_i^\alpha} \frac{\partial m}{\partial X_j^\beta} - \frac{\partial\phi}{\partial X_j^\beta} \frac{\partial m}{\partial X_i^\alpha} \right\}; \quad (8)$$

∇^α is the gradient with respect to X^α , and E is the energy of the system.

3 APPLICATION TO AN N-VORTEX SYSTEM

We now consider explicitly a classical two-dimensional easy-plane Heisenberg ferromagnet where the localized excitations are vortices. In the continuum limit (derived from a square lattice) the Hamiltonian has the form

$$\mathcal{H} = \frac{JS^2}{2} \int d^2r \{ (\text{grad } \mathbf{S})^2 + 4\delta S_z^2 - \delta (\text{grad } S_z)^2 \}; \quad (9)$$

where δ measures the anisotropy of the system ($\delta = 0$ corresponds to the isotropic, $\delta = 1$ to the planar Heisenberg model). As was shown earlier⁷ there exist two distinct types of vortices which are stable for different values of the anisotropy parameter δ : While for large values of the anisotropy the vortex spin field is confined mainly to the XY-plane, in the following discussion we only consider the regime $0 < \delta \lesssim 0.3$, where already the static vortices exhibit a well-localized out-of-plane structure (Fig. 1a).

The energy of such a N-vortex system is given by⁸

$$E = - \sum_{\alpha > \beta} k q^\alpha q^\beta \ln |\mathbf{X}^\alpha - \mathbf{X}^\beta|, \quad k = 2\pi JS^2. \quad (10)$$

from which we derive

$$\nabla^\alpha E = -kq^\alpha \sum_{\beta \neq \alpha} q^\beta \frac{\mathbf{X}^\alpha - \mathbf{X}^\beta}{|\mathbf{X}^\alpha - \mathbf{X}^\beta|^2}. \quad (11)$$

To calculate the tensors $\mathbf{G}^{\alpha\beta}$ and $\mathbf{M}^{\alpha\beta}$ we need explicit expressions for the fields $m(\mathbf{r})$ and $\phi(\mathbf{r})$. As far as we are aware there have so far been derived only single vortex solutions which are for a vortex at position \mathbf{X} , up to lowest order in the velocity⁷,

$$\phi(\mathbf{r}; \mathbf{X}; \dot{\mathbf{X}}) = q \arctan \frac{y - X_2}{x - X_1} + p \tilde{\phi}(\mathbf{r} - \mathbf{X}) \frac{(x - X_1)\dot{X}_1 - (y - X_2)\dot{X}_2}{|\mathbf{r} - \mathbf{X}|}, \quad (12)$$

$$m(\mathbf{r}; \mathbf{X}; \dot{\mathbf{X}}) = pm_0(|\mathbf{r} - \mathbf{X}|) + qC\tilde{m}(\mathbf{r} - \mathbf{X}) \frac{(x - X_1)\dot{X}_2 - (y - X_2)\dot{X}_1}{|\mathbf{r} - \mathbf{X}|^2}.$$

The numbers $q = \pm 1$ and $p = \pm 1$ are the vorticity and the sign of the static out-of-plane structure, respectively, and $C = 1/4JS\delta$. For the calculation of the main contributions of $\mathbf{G}^{\alpha\beta}$ we need only the static parts of $\phi(\mathbf{r})$ and $m(\mathbf{r})$, while for the calculation of $\mathbf{M}^{\alpha\beta}$ it is sufficient to include the velocity dependent part of $m(\mathbf{r})$ (the

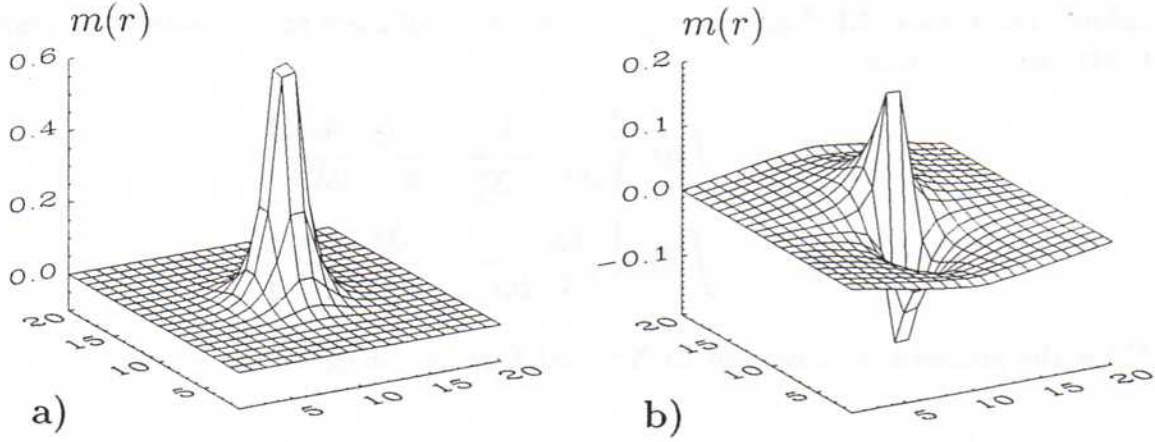


Fig. 1: Out-of-plane fields of single vortices: a) static structure ($\delta = 0.2$); b) velocity-induced structure for a vortex moving in the x -direction ($\delta = 1.0$).

equivalent part of the $\phi(\mathbf{r})$ -field is already exponentially small). The function $m_0(r)$ has, in the δ -range under consideration, the asymptotic form

$$m_0(r) \cong \sqrt{r_v/r} e^{-r/r_v}, \quad r \gg 1, \quad (13)$$

where $r_v = \sqrt{(1-\delta)/\delta}$ is the vortex core radius. For the function $\tilde{m}(r)$ we only know that it approaches 1 for $r \gg 1$. However, for small r the complete velocity-dependent out-of-plane structure shows approximately the same angular dependence as for large r (cf. Fig. 1b), and we will assume in our calculation that $\tilde{m}(r)$ is a function of the radial coordinate r only.

If we now consider only those systems for which the vortices are all well separated from each other (i.e. the mutual distances are larger than $\max(r_v, c)$), we can assume that the total N -vortex fields $\phi(\mathbf{r})$ and $m(\mathbf{r})$ can be well approximated by a linear superposition of single-vortex solutions. Using this ansatz we obtain

$$\mathbf{M}_{ij}^{\alpha\beta} = \pi C \begin{cases} (\ln(L/d^{\alpha\beta}) + \psi(d^{\alpha\beta}))\delta_{ij} & \alpha = \beta \\ q^\alpha q^\beta \left[\ln(L/d^{\alpha\beta})\delta_{ij} - \chi(d^{\alpha\beta}) \left(\delta_{ij} - 2\hat{\mathbf{P}}_{ij}^{\alpha\beta} \right) \right] & \alpha \neq \beta \end{cases} \quad (14)$$

$$\mathbf{G}_{ij}^{\alpha\beta} = 2\pi \begin{cases} q^\alpha p^\alpha \epsilon_{ij} & \alpha = \beta \\ q^\alpha p^\beta \left[\Delta_+^{\alpha\beta} m_0(d^{\alpha\beta})\epsilon_{ij} - \Delta_-^{\alpha\beta} \zeta(d^{\alpha\beta}) \left(\epsilon_{ij} - 2\epsilon_{ik} \hat{\mathbf{S}}_{kj}^{\alpha\beta} \right) \right] & \alpha \neq \beta \end{cases} \quad (15)$$

where δ_{ij} and ϵ_{ij} are the unit tensor and the completely antisymmetric tensor in two dimensions, respectively, L is an upper cut-off for the system size, and $\Delta_+^{\alpha\beta}$ ($\Delta_-^{\alpha\beta}$) is 1 (0) for $q^\alpha p^\beta = q^\beta p^\alpha$ and 0 (1) for $q^\alpha p^\beta = -q^\beta p^\alpha$. The action of the two operators $\hat{\mathbf{P}}_{ij}^{\alpha\beta}$ and $\hat{\mathbf{S}}_{ij}^{\alpha\beta}$ on a vector is to project out its components parallel and perpendicular to $\mathbf{X}^\alpha - \mathbf{X}^\beta$, respectively. Additionally, we have introduced the abbreviations

$$\begin{aligned} d^{\alpha\beta} &= |\mathbf{X}^\alpha - \mathbf{X}^\beta|, & \zeta(d) &= 2d^{-2} \int_{a_0}^d dr r m_0(r), \\ \chi(d) &= d^{-2} \int_{a_0}^d dr r \tilde{m}(r), \quad \text{and} & \psi(d) &= \int_{a_0}^d dr \tilde{m}(r)/r, \end{aligned} \quad (16)$$

where a_0 is a lower cut-off of the order of the lattice constant.

All the masses $M^{\alpha\beta}$ depend logarithmically on the system size. However, as we will show in the next section, there can exist dynamical degrees of freedom with an associated mass which will remain finite, even on an infinite system. Moreover, Eqn. (14) shows that the masses depend on the vorticities q^α only. The gyrovectors, in contrast, are always finite and are characterized by products of the form $q^\alpha p^\beta$. In particular, if $q^\alpha p^\beta = q^\beta p^\alpha$ the two-vortex contributions are exponentially small, while they become more long-ranged (i.e. they decrease with $(d^{\alpha\beta})^{-2}$) for $q^\alpha p^\beta = -q^\beta p^\alpha$.

4 DISCUSSION OF A 2-VORTEX SYSTEM

For a two-vortex system it is more convenient to rewrite the equations of motion in the variables $\mathbf{R} = \frac{1}{2}(\mathbf{X}^1 + \mathbf{X}^2)$ and $\mathbf{Y} = \mathbf{X}^1 - \mathbf{X}^2$. With the new masses

$$\begin{aligned}\bar{\mathbf{M}} &= \mathbf{M}^{11} + \mathbf{M}^{12} = \mathbf{M}^{22} + \mathbf{M}^{21} \\ \tilde{\mathbf{M}} &= \frac{1}{2}(\mathbf{M}^{11} - \mathbf{M}^{12}) = \frac{1}{2}(\mathbf{M}^{22} - \mathbf{M}^{21})\end{aligned}\tag{17}$$

and gyrotensors

$$\begin{aligned}\bar{\mathbf{G}}^i &= \mathbf{G}^{ii} + \mathbf{G}^{ij} \\ \tilde{\mathbf{G}}^i &= \frac{1}{2}(\mathbf{G}^{ii} - \mathbf{G}^{ij}), \quad i, j = 1, 2, \quad i \neq j,\end{aligned}\tag{18}$$

we arrive at the two equations of motion

$$\begin{aligned}\bar{\mathbf{M}}\ddot{\mathbf{R}} + \tilde{\mathbf{M}}\ddot{\mathbf{Y}} + \bar{\mathbf{G}}^1\dot{\mathbf{R}} + \tilde{\mathbf{G}}^1\dot{\mathbf{Y}} &= \frac{kq^1q^2}{Y^2}\mathbf{Y} \\ \bar{\mathbf{M}}\ddot{\mathbf{R}} - \tilde{\mathbf{M}}\ddot{\mathbf{Y}} + \bar{\mathbf{G}}^2\dot{\mathbf{R}} - \tilde{\mathbf{G}}^2\dot{\mathbf{Y}} &= -\frac{kq^1q^2}{Y^2}\mathbf{Y}.\end{aligned}\tag{19}$$

In an appropriate coordinate system with one axis parallel, the other perpendicular, to \mathbf{Y} the mass tensors become diagonal:

$$\bar{\mathbf{M}}_{ij} = \begin{pmatrix} \bar{M}_1 & 0 \\ 0 & \bar{M}_2 \end{pmatrix}, \quad \tilde{\mathbf{M}}_{ij} = \frac{1}{2} \begin{pmatrix} \tilde{M}_1 & 0 \\ 0 & \tilde{M}_2 \end{pmatrix},\tag{20}$$

with

$$\begin{aligned}\bar{M}_1 &= \pi C [(1 + q^1q^2) \ln(L/d)/2 + (\psi + q^1q^2\chi)] \\ \bar{M}_2 &= \pi C [(1 + q^1q^2) \ln(L/d)/2 + (\psi - q^1q^2\chi)] \\ \tilde{M}_1 &= \pi C [(1 - q^1q^2) \ln(L/d)/2 + (\psi - q^1q^2\chi)] \\ \tilde{M}_2 &= \pi C [(1 - q^1q^2) \ln(L/d)/2 + (\psi + q^1q^2\chi)].\end{aligned}\tag{21}$$

The dependence of the gyro tensors on the products $q^\alpha p^\beta$ suggests a division of the following discussion into the two parts:

a) $q^1p^2 = q^2p^1$: For this case we find for the gyro tensors

$$\begin{aligned}\bar{\mathbf{G}}^1_{ij} &= \bar{\mathbf{G}}^2_{ij} \equiv \bar{\mathbf{G}}_{ij} = 2\pi q^1(p^1 + p^2m_0)\epsilon_{ij} \\ \tilde{\mathbf{G}}^1_{ij} &= \tilde{\mathbf{G}}^2_{ij} \equiv \tilde{\mathbf{G}}_{ij} = \pi q^1(p^1 - p^2m_0)\epsilon_{ij} \equiv G\epsilon_{ij}.\end{aligned}\tag{22}$$

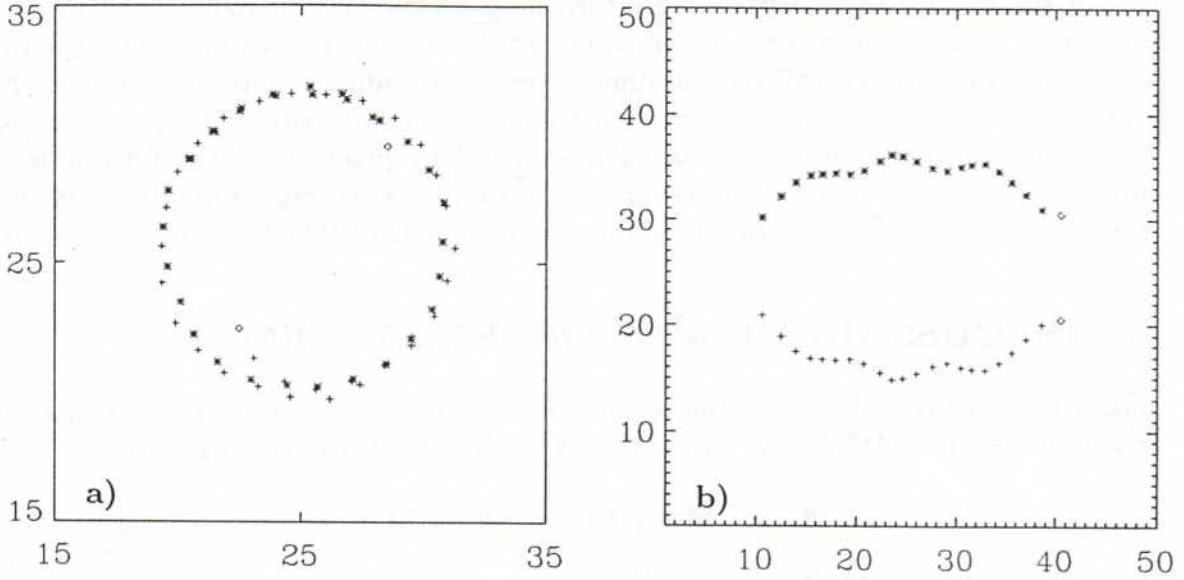


Fig. 2: Vortex-vortex simulations on a 50×50 square lattice; a) $q^1 p^2 = q^2 p^1$; b) $q^1 p^2 = -q^2 p^1$. \diamond : initial positions. We used a small damping for the first 100 integration steps to remove initially excited spinwaves.

Inserting this result into Eqns. (19) it follows that the center-of-mass (cms) coordinate \mathbf{R} is constant (which, without loss of generality, can be set to zero), and there remains a single equation for the relative distance \mathbf{Y} . Since $\tilde{\mathbf{G}}$ is a completely antisymmetric tensor we can introduce the gyrovector $\mathbf{g} = G\mathbf{e}_z$ to finally obtain

$$\tilde{\mathbf{M}}\ddot{\mathbf{Y}} - \mathbf{g} \times \dot{\mathbf{Y}} = \frac{kq^1 q^2}{Y^2} \mathbf{Y}. \quad (23)$$

Eqn. (23) describes a rotation of the two vortices around each other at distance d with angular velocity

$$\omega_0 = \frac{1}{2} \frac{G}{\tilde{M}_1} \left\{ 1 - \sqrt{1 - \frac{4q^1 q^2 \tilde{M}_1}{\pi d^2}} \right\} \quad (24)$$

and an additional cyclotron-like oscillation around this trajectory with frequency

$$\omega^2 = \omega_c^2 \left\{ 1 - \frac{2q^1 q^2 \tilde{M}_2}{\pi d^2} + 2 \frac{\tilde{M}_2 - \tilde{M}_1 \omega_0}{\sqrt{\tilde{M}_1 \tilde{M}_2} \bar{\omega}_c} - \frac{d}{2} \frac{\partial \ln \tilde{M}_1}{\partial d} \left(\frac{\omega_0}{\bar{\omega}_c} \right)^2 \right\} \quad (25)$$

with $\omega_c^2 = G^2 / \tilde{M}_1 \tilde{M}_2$. For the case $q^1 q^2 = +1$ both masses \tilde{M}_1 and \tilde{M}_2 are independent of the system size. Since ω is inversely proportional to the masses this leads to a high cyclotron frequency with small amplitudes which are of the order of the lattice discreteness scale (Fig. 2a). For $q^1 q^2 = -1$, however, the masses increase logarithmically with the system size. These large masses lead to small values of ω and the cyclotron oscillations are easily observed in numerical simulations (see Fig. 1 in Ref. 9). The two terms in (25) which are due to the different values of the masses \tilde{M}_1 , \tilde{M}_2 and the dependence of the masses on the mutual vortex distance lead to corrections which are consistent with the observed d -dependence of the frequency ω .

