Foundations of QM. Need QM when wave properties are apparent, $\lambda = \frac{h}{p} \simeq$ size of "system". Understand wave/particle duality, Compton, photoelectric effects, correspondence principle and Bohr-Sommerfeld quantization: $\oint p \, dq = nh$. Connections between particle and wave properties: $E = \hbar\omega, \vec{p} = \hbar\vec{k} = \frac{h}{\lambda}\hat{k}$.

CM $\rightarrow$ QM. Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t)$ and Hamiltonian $H(\{q_i\}, \{p_i\}, t)$, coordinates q_i and canonically conjugate momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$. Coupling to electromagnetic fields via $H = \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2 + q\phi$.

Heisenberg Uncertainty Principle. Relations $\Delta x \Delta p_x \geq \frac{\hbar}{2}$, as well as between ΔE and Δt, and their interpretation. How to define ΔA for any observable.

Wavepackets. $\Psi(\vec{r}, t)$ or $\Phi(\vec{p}, t)$, with $\Phi(\vec{p}, t) = (2\pi\hbar)^{-3/2} \int d^3r \; \Psi(\vec{r}, t) e^{-i\vec{p}\cdot\vec{r}/\hbar}$. Uncertainties in x, p_x, etc., spreading with time when $t \geq \frac{\Delta x}{\Delta v_x}$.

Probabilisitic Interpretation. $|\Psi(\vec{r}, t)|^2 d^3r$ is the relative probability to find the "particle" in volume d^3r at time t. $|\Phi(\vec{p}, t)|^2 d^3p$ is the relative probability to find the "particle" in momentum volume d^3p at time t.

Operators and Schrodinger Equations. Formulas like momentum $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$ in coordinate representation, and position $\hat{x} = +i\hbar\frac{\partial}{\partial p_x}$ in momentum representation. Energy operator $\hat{E} = i\hbar\frac{\partial}{\partial t}$. Can write Schrodinger equation $H\Psi = E\Psi$ for a given Hamiltonian, in either the coordinate or momentum representation.

Expectations and Dynamics. Meaning of $\langle A \rangle = \langle \Psi, A\Psi \rangle = \frac{\int dq \;\; \Psi^*(q) A \Psi(q)}{\int dq \;\; \Psi^*(q)\Psi(q)}$, variance $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$. Dynamics via $\frac{d}{dt}\langle A \rangle = \frac{1}{i\hbar}\langle [A, H] \rangle + \langle \frac{\partial A}{\partial t} \rangle$. Algebra of commutators, fundamental one being $[x, p_x] = i\hbar$. Exciting relations like expansion: $[AB, C] = A[B, C] + [A, C]B$, and Poisson-bracket-like: $[p_x, F] = -i\hbar\frac{\partial}{\partial x}F$, $[x, G] = i\hbar\frac{\partial}{\partial p_x}G$.

Eigenvalue Problems. EV Eqn. $Au_n(q) = a_n u_n(q)$, means ΔA=0 when system is in an eigenstate u_n of A. Eigenspectrum may be *discrete* or *continuous*. Normalize continuum states by $L \rightarrow \infty$ limit of finite system discrete eigenstates: $u(a; q) = \sqrt{\rho(a)} \; u_n(q)$. Density of states or density of eigenvalues is number of quantum states per unit range of the observable a, $\rho(a) = \frac{\Delta n}{\Delta a}$.

Expanding Arbitrary Ψ in Eigenstates. Consider measuring an observable A. Expand current wavefunction $\Psi(q)$ of system in orthonormal eigenstates (u_n) of A: $\Psi(q) = \sum_n c_n u_n(q)$, where the probability amplitudes are $c_n = \langle u_n, \Psi \rangle = \int \, dq \; u_n^*(q)\Psi(q)$.

Measurements and Probabilities. If A is measured in an experiment, individual measurements can only give one of the eigenvalues, a_n. The probability that the measured result is a_n is $\mathcal{P}_n = \frac{|c_n|^2}{\sum_m |c_m|^2}$. If original $\Psi(q)$ is normalized already, $\langle \Psi, \Psi \rangle = \sum_m |c_m|^2 = 1$. Then the probabilities are $\mathcal{P}_n = |c_n|^2$, the squares of probability amplitudes.

Expectation Values Again. An expansion of $\Psi(q)$ in eigenstates of A makes expectation values easier, using the probabilities of the various eigenvalues: $\langle A \rangle \equiv \langle \Psi, A\Psi \rangle = \sum_n \mathcal{P}_n a_n$. If Ψ has unit normalization, $\langle A \rangle = \sum_n |c_n|^2 a_n$. Similarly, have $\langle A^2 \rangle = \sum_n |c_n|^2 a_n^2$, etc.

Why Commutators are Fun. 1) Dynamics: $\frac{d}{dt}\langle A \rangle = \frac{1}{i\hbar}\langle [A, H] \rangle + \langle \frac{\partial A}{\partial t} \rangle$. 2) Conservation laws: $[A, H] = 0$ implies observable A is conserved. 3) Uncertainty laws: $[x, p_x] = i\hbar$ is due to Heisenberg uncertainty principle for incompatible observables. 4) Compatible observables: They commute ($[A, B] = 0$) and their measurements do not affect each other. $[A, B] = 0$ implies they have a complete orthonormal set of eigenfunctions in common, and conversely.

Expanding Ψ in Continuum Eigenstates. If measured observable A has a **continuous** spectrum, $Au(a;q) = au(a;q)$, then we expand wavefunction $\Psi(q) = \int da \; c(a)u(a;q)$, where $c(a) = \langle u(a;q), \Psi \rangle = \int dq \; u^*(a;q)\Psi(q)$. When Ψ is unit normalized, so is $c(a)$: $\langle \Psi, \Psi \rangle = \langle c(a), c(a) \rangle = 1$, or $\int dq \; \Psi^*(q)\Psi(q) = \int da \; c^*(a)c(a) = 1$.

The function $c(a)$ can be considered as the wavefunction in the A-representation. It could be obtained from the $L \to \infty$ limit of discrete probability amplitude c_n by $c(a) = \sqrt{\rho(a)} \; c_n$, using the density of states.

Expectations/Probabilities for Continuum Spectrum. For a normalized state: Just as $|\Psi(q)|^2 dq$ is the probability of finding the system between q and $q + dq$ (a measurement), then alternatively, $|c(a)|^2 da$ is the probability of measuring the system to have eigenvalue of operator A between a and $a + da$. The probability density is $\mathcal{P}(a) = |c(a)|^2$. Expectations may be simplified by $\langle A \rangle = \langle \Psi, A\Psi \rangle = \int da \; |c(a)|^2 a$ or more generally, $\langle F(A) \rangle = \langle \Psi, F(A)\Psi \rangle = \int da \; |c(a)|^2 F(a)$.