Electrodynamics II Exam 1. Part A (120 pts.) Closed Book Radiation & Scattering

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Instructions: Use SI units. Only brief derivations here. State your responses clearly, and define your variables in words. Write on other side if needed.

1. (10) Radiation is being produced by a localized time-dependent source with $\rho(\mathbf{x},t), \mathbf{J}(\mathbf{x},t)$. Write the wave equation for the vector potential $\mathbf{A}(\mathbf{x},t)$ that is produced outside the source.

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -4\pi\vec{f} \qquad \text{where } \vec{f} = \frac{\mu_0}{4\pi}\vec{J}(\vec{x},t) \text{ is the source function.}$$

2. (10) Give an expression for the retarded Green's function that applies to the wave equation of the previous question.

$$G(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left[t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right]$$

delay time for signal to travel from $\vec{x}'$ to $\vec{x}$.

which applies itself at the point $t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$

3. (10) If the sources have harmonic time dependence $\sim e^{-i\omega t}$, what expression results for the vector potential $\mathbf{A}(\mathbf{x},t)$ in a general case (without approximations)?

Use $\vec{J}(\vec{x}',t') = \vec{J}(\vec{x}')e^{-i\omega t'}$, $\quad \vec{A}(\vec{x},t) = \int d^3x' \int dt'\, G(\vec{x},t;\vec{x}',t')\,\frac{\mu_0}{4\pi}\vec{J}(\vec{x}',t')$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}')\,e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}\,e^{-i\omega t}, \quad \text{suggests } k=\frac{\omega}{c}.$$

4. (10) If the source is a harmonically oscillating electric dipole $\mathbf{p}(t) = \mathbf{p}e^{-i\omega t}$, with $\mathbf{p} = \int d^3x\, \mathbf{x}\rho$, what results for $\mathbf{A}(\mathbf{x},t)$ very far away?

Expand $|\vec{x}-\vec{x}'| \approx r - \hat{n}\cdot\vec{x}' \approx r$ and $\int d^3x'\,\vec{J}(\vec{x}') \to -\int d^3x'\,\vec{x}'\underbrace{(\vec{\nabla}'\cdot\vec{J}')}_{i\omega\rho'} \to -i\omega\vec{p}$

by $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \;\to$

$$\vec{A}(\vec{x},t) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega\vec{p})\,e^{-i\omega t}$$

$$\text{or } \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega\vec{p}).$$

5. (10) For a harmonically oscillating electric dipole, show how an expression for the magnetic field $\mathbf{H}$ very far away is obtained.

Use $\vec{H} = \frac{1}{\mu_0}\vec{B} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} \simeq \frac{1}{\mu_0}\,ik\,\hat{n}\times\vec{A}$

$$\vec{H} \simeq \frac{ik}{\mu_0}\,\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\,\hat{n}\times\vec{p} = \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}\,\hat{n}\times\vec{p}.$$

$\omega = ck$

6. (10) If a radiation source is producing magnetic field $\mathbf{H}(\mathbf{x},t)$ outside the source, how is the electric field $\mathbf{E}(\mathbf{x},t)$ obtained?

From Ampere/Maxwell, $\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} = -i\omega\epsilon_0\vec{E} \;\Rightarrow\; \vec{E} = \frac{i}{\omega\epsilon_0}\vec{\nabla}\times\vec{H}$

$$\text{or } \vec{E} = \frac{i}{c\epsilon_0 k}\vec{\nabla}\times\vec{H} = \frac{i}{k}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\vec{\nabla}\times\vec{H} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}.$$

$\frac{1}{\sqrt{\mu_0\epsilon_0}} = c$

7. (10) A radiation source produces fields **E** and **H** far from the source. How do you use them to obtain the power radiated per unit solid angle, along a direction $\hat{\mathbf{n}}$?

$$\frac{dP}{d\Omega} = r^2 \hat{n} \cdot \vec{S} = r^2 \hat{n} \cdot (\vec{E} \times \vec{H}) \quad \text{(instantaneous)}$$

or $$\frac{dP}{d\Omega} = \frac{1}{2} r^2 \hat{n} \cdot \mathrm{Re}(\vec{E}^* \times \vec{H}) \quad \text{(time-averaged)}$$

8. (10) What is the definition of a "vector spherical harmonic?" List one orthogonality property that they have.

$$\vec{X}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \vec{\mathcal{L}} Y_{\ell m}$$ where $\vec{\mathcal{L}} = \vec{r} \times (-i\vec{\nabla})$ is the angular momentum operator

$$\int d\Omega \, \vec{X}^*_{\ell m} \cdot \vec{X}_{\ell' m'} = \delta_{\ell\ell'} \delta_{mm'}.$$

9. (10) Write out the general solution of the source-free scalar Helmholtz equation, $(\nabla^2 + k^2)\psi(\mathbf{x}, \omega) = 0$, in spherical coordinates.

$$\psi(\vec{x}, \omega) = \sum_{\ell m} \left[A_{\ell m} h_\ell^{(1)}(kr) + B_{\ell m} h_\ell^{(2)}(kr) \right] Y_{\ell m}(\theta, \phi)$$

spherical Bessel functions

10. (10) Light of wave vector $\mathbf{k}_0 = k\hat{\mathbf{n}}_0$ and polarization $\hat{\epsilon}_0$ is scattered into a new wave vector $\mathbf{k} = k\hat{\mathbf{n}}$ and polarization $\hat{\epsilon}$. Write an expression for the differential cross section $d\sigma/d\Omega$ for this process.

$$\frac{d\sigma}{d\Omega} = \frac{dP/d\Omega}{|\vec{S}_{inc}|} \rightarrow \frac{r^2 |\hat{\epsilon}^* \cdot \vec{E}_{scatt}|^2}{|\hat{\epsilon}_0^* \cdot \vec{E}_{inc}|^2}$$ using scattered and incident fields.

11. (10) Describe how the total scattering cross section of a nonconducting dieletric sphere of permittivity ε varies with its radius a and the wavelength λ of the light.

$$\sigma \propto k^4 V^2 \propto \left(\frac{2\pi}{\lambda}\right)^4 a^6 \sim \frac{a^6}{\lambda^4}.$$

12. (10) Describe how the total cross section (scattering plus absorption) determines the extinction (or attentuation) coefficient α for a medium.

If σ_1 is total cross section of one particle, and there are N particles per unit volume, then $\alpha = N\sigma_1$ is the extinction coefficient.

Part A Score = ________/120

Electrodynamics II Exam 1. Part B (130 pts.) Open Book Radiation & Scattering

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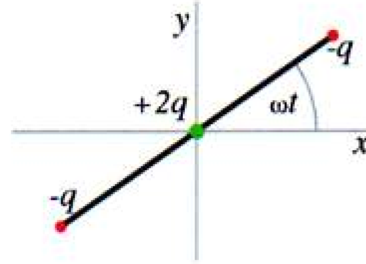
Instructions: Use SI units. Please show the details of your derivations here. Explain your reasoning for full credit. Open-book only, no notes.

1. (130) Charges $-q$, $-q$ and $+2q$ are fixed at the two ends and center, respectively, of a rod of length $2a$ that rotates in the xy-plane at angular speed ω around the z-axis, forming a rotating electric quadrupole. The position of the charge at one end can be written $x = a\cos\omega t$, $y = a\sin\omega t$; the other is directly opposite this point.



a) (20) Write out the time-dependent charge density $\rho(\mathbf{x}, t)$ in terms of delta functions in Cartesian coordinates.

b) (40) Determine the nonzero Cartesian components of its time-dependent electric quadrupole tensor $\mathbf{Q}(t)$. Summarize that result as a 3×3 matrix.

c) (20) Express the result for $\mathbf{Q}(t)$ in complex form with a harmonic time dependence. What is the frequency of the oscillating part? What will be the frequency of the radiation it produces? What is the wavelength?

d) (20) Find the radiated magnetic field $\mathbf{H}$ in the radiation zone. Give the xyz components of $\mathbf{H}$ as functions of the angular direction θ, ϕ of unit wave vector $\hat{\mathbf{n}}$.

e) (10) At a point on the y-axis at radius $r \gg \lambda$, what are the directions of the magnetic and electric field vectors?

f) (20) Bonus. Find the formula for the time-averaged power radiated per unit solid angle, $dP/d\Omega$, in the far field, as a function of spherical angles θ, ϕ.

a) Use a delta function at each charge position.

$$\rho(\vec{x},t) = 2q\,\delta(\vec{x}) - q\,\delta(x-a\cos\omega t)\,\delta(y-a\sin\omega t)\,\delta(z) - q\,\delta(x+a\cos\omega t)\,\delta(y+a\sin\omega t)\,\delta(z)$$

also where $\delta(\vec{x}) = \delta(x)\,\delta(y)\,\delta(z)$.

Obviously as time progresses the points where the deltas are nonzero rotate around.

b) Quadrupole tensor is $Q_{\alpha\beta} = \int d^3x\ \rho(\vec{x},t)\,(3x_\alpha x_\beta - r^2\delta_{\alpha\beta})$.

One can expect nonzero values of Q_{xx}, Q_{yy}, Q_{zz}, and Q_{xy}, Q_{yx}.

$$Q_{xx} = \int d^3x \left[2q\,\delta(\vec{x}) - q\,\delta(x-a\cos\omega t)\delta(y-a\sin\omega t)\delta(z) - q\,\delta(x+a\cos\omega t)\delta(y+a\sin\omega t)\delta(z)\right]$$
$$\times\ (3x^2 - x^2 - y^2 - z^2)$$

All $\delta(z)$ force $z=0$. The $2q$ charge also will give zero in this integral. The two negative charges contribute equal amounts, due to squares.

Effectively, $3x^2 - r^2 \to 2x^2 - y^2 \to 2a^2\cos^2\omega t - a^2\sin^2\omega t$

$$Q_{xx} = -2qa^2\left[2\cos^2\omega t - \sin^2\omega t\right] = -2qa^2(3\cos^2\omega t - 1)$$

use an identity like $\cos 2\omega t = 2\cos^2\omega t - 1$ or $\cos^2\omega t = \frac{1}{2}(1+\cos 2\omega t)$

$$Q_{xx} = -2qa^2\left[3\times\tfrac{1}{2}(1+\cos 2\omega t) - 1\right] = -2qa^2\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2\omega t\right)$$

For Q_{yy}, will have $3y^2 - r^2 \to 2y^2 - x^2 \to 2a^2\sin^2\omega t - a^2\cos^2\omega t$.

$$Q_{yy} = -2qa^2\left[2\sin^2\omega t - \cos^2\omega t\right] = -2qa^2\left[2 - 3\cos^2\omega t\right]$$

$$Q_{yy} = -2qa^2\left[2 - \tfrac{3}{2}(1+\cos 2\omega t)\right] = -2qa^2\left(\tfrac{1}{2} - \tfrac{3}{2}\cos 2\omega t\right).$$

For Q_{zz}, will have $3z^2 - r^2 \to 2z^2 - x^2 - y^2 \to -x^2 - y^2 \to -a^2$.

$$Q_{zz} = -2q(-a^2) = +2qa^2$$

Also
$$Q_{xy} = Q_{yx} = -2qa^2\cdot(3)\cos\omega t\sin\omega t = -2qa^2\times\tfrac{3}{2}\sin 2\omega t$$

c) Have found

$$\overleftrightarrow{Q}(t) = 2qa^2 \begin{pmatrix} -\frac{1}{2} - \frac{3}{2}\cos 2\omega t & -\frac{3}{2}\sin 2\omega t & 0 \\ -\frac{3}{2}\sin 2\omega t & -\frac{1}{2} + \frac{3}{2}\cos 2\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using $\cos 2\omega t = Re(e^{-i2\omega t})$, $\sin 2\omega t = Re(i\, e^{-i2\omega t})$

$$\overleftrightarrow{Q}(t) = qa^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + Re\left\{ 3qa^2 \begin{pmatrix} -1 & -i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{i2\omega t} \right\}$$

The second part is the oscillating part, at frequency 2ω.

Simplify it as complex part $\rightarrow$ $\overleftrightarrow{Q} = 3qa^2 \begin{pmatrix} -1 & -i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Let $Q_0 = 3qa^2$.

The wavelength produced is found from $k = \frac{2\pi}{\lambda} = \frac{2\omega}{c}$.

or $\lambda = \frac{\pi c}{\omega}$.

d) Rad. field is $\vec{H} = \frac{-ick^3}{24\pi} \frac{e^{ikr}}{r} \hat{n} \times \vec{Q}(\hat{n})$.

where $\vec{Q}(\hat{n}) = \overleftrightarrow{Q} \cdot \hat{n} = \sum_{\alpha\beta} \hat{e}_\alpha Q_{\alpha\beta} n_\beta$; $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

e.g. $Q_x = Q_{xx} n_x + Q_{xy} n_y = -Q_0 \sin\theta\cos\phi - iQ_0 \sin\theta \sin\phi$

$= -Q_0 \sin\theta (\cos\phi + i \sin\phi) = -Q_0 \sin\theta \, e^{i\phi}$

$Q_y = Q_{yy} n_y + Q_{yx} n_x = Q_0 \sin\theta \sin\phi - iQ_0 \sin\theta\cos\phi$

$= -iQ_0 \sin\theta (\cos\phi + i\sin\phi) = -iQ_0 \sin\theta \, e^{i\phi}$.

$Q_z = Q_{zz} n_z = 0$.

or $\vec{Q} = -Q_0 \sin\theta (\hat{x} + i\hat{y}) e^{i\phi}$

$\hat{n} \times \vec{Q} = -(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta) \times (\hat{x} + i\hat{y}) Q_0 \sin\theta \, e^{i\phi}$

$= -\left[\hat{x}(-i\cos\theta) + \hat{y}(\cos\theta) + \hat{z}(i\sin\theta\cos\phi - \sin\theta\sin\phi)\right] Q_0 \sin\theta \, e^{i\phi}$

$= -\left[(-i\hat{x} + \hat{y}) \cos\theta + i\hat{z} \sin\theta \, e^{i\phi}\right] Q_0 \sin\theta \, e^{i\phi}$

The components of $\vec{H}$ are then $\left(\frac{-ick^3}{24\pi} \frac{e^{ikr}}{r}\right)$ times this result.

where $Q_0 = 3qa^2$.

or state as

$\hat{n} \times \vec{Q} = iQ_0 \left[(\hat{x} + i\hat{y}) \cos\theta - \hat{z} \sin\theta \, e^{i\phi}\right] \sin\theta \, e^{i\phi}$

e) At a pt on y-axis we have $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$ and $\cos\theta = 0$.

$$\hat{n}\times\vec{Q} = -i\hat{z}\,Q_0 \sin^2\theta\, e^{2i\phi} = -i\hat{z}Q_0 e^{i\pi} = iQ_0\hat{z}.$$

The $\vec{H}$-vector is oscillating in the $\hat{z}$ direction.
The radial direction is $\hat{y}$. From $\vec{E} = Z_0\,\hat{n}\times\vec{H}$, $\hat{n}=\hat{y}$,
then the $\vec{E}$-vector must be oscillating in the $-\hat{x}$-direction.

f) $\frac{dP}{d\Omega} = \frac{1}{2} r^2 \operatorname{Re}\left(\hat{n}\cdot(\vec{E}^*\times\vec{H})\right) = \frac{1}{2} r^2 Z_0 |\vec{H}|^2$, is easiest to apply.

$$\frac{dP}{d\Omega} = \frac{1}{2} Z_0 \left(\frac{ck^3}{24\pi}\right)^2 Q_0^2 \left|\left[(i\hat{x}-\hat{y})\cos\theta - i\hat{z}\sin\theta\, e^{i\phi}\right]\sin\theta\, e^{i\phi}\right|^2$$

$$= \frac{Z_0}{2}\left(\frac{ck^3}{24\pi}\right)^2 Q_0^2\left[\cos^2\theta + \cos^2\theta + \sin^2\theta\right]\sin^2\theta$$

$$= \frac{Z_0}{2}\left(\frac{ck^3 Q_0}{24\pi}\right)^2 (1+\cos^2\theta)(1-\cos^2\theta) = \frac{Z_0}{2}\left(\frac{ck^3 Q_0}{24\pi}\right)^2(1-\cos^4\theta).$$

There is no dependence on ϕ – azimuthal symmetry.

Could also get the total power, $P = \int d\Omega\, \frac{dP}{d\Omega} = 2\pi\int_{-1}^{1} d(\cos\theta)\,\frac{dP}{d\Omega}$

Which involves $\int_{-1}^{1} d(\cos\theta)(1-\cos^4\theta) = \cos\theta - \frac{1}{5}\cos^5\theta\Big|_{-1}^{1} = \frac{8}{5}$.

$$P = \frac{Z_0}{2}\left(\frac{ck^3 Q_0}{24\pi}\right)^2 (2\pi)\left(\frac{8}{5}\right) = \frac{8\pi}{5} Z_0\left(\frac{ck^3 Q_0}{24\pi}\right)^2 \qquad Q_0 = 3qa^2.$$