Electrodynamics II. Exam 3. Part A (150 pts.)
Relativistic Particles, Fields and Radiation Closed Book

Name______________________________________ KSU 2015/05/11 14:00 - 15:50

Instructions: Some small derivations here, state your responses clearly, define your variables in words if they are not common usage.

1. A particle with charge e, mass m is moving with coordinate $\mathbf{r}(t)$ and velocity $\mathbf{u}(t)$ in some EM field defined in a given inertial frame of reference by scalar and vector potentials $\Phi, \mathbf{A}$.

    a. (10) When the particle is moving nonrelativistically, what is the simplest Lagrangian that describes its dynamics?

    b. (10) Now consider the particle with relativistic motion. Write out the full relativistic Lagrangian needed in the the given inertial frame.

    c. (10) For the particle with relativistic motion, use your previous answer to get the canonical momentum $\mathbf{P}$, in terms of the mechanical momentum $\mathbf{p} = \gamma m \mathbf{u}$.

2. (8) In classical mechanics for periodic motion there are action integrals of the form $J_i = \oint p_i \, dq_i$ that are considered adiabatic quantities (also, they become quantized in amounts nh). What adiabatic invariant for charged particle motion results from this?

3. A particle (mass m charge e) moves in a uniform constant magnetic field $\mathbf{B} = B\hat{z}$.

   a. (8) Calculate the angular frequency ω of its cylclotron motion.

   b. (8) For a positive charge e, determine the sense of rotation around lines of $\mathbf{B}$ on a diagram.

4. Consider the Lagrangian for EM fields.

   a. (8) Write a Lagrangian density $\mathcal{L}$ for electromagnetic fields generated by the 4-vector current density J_α.

   b. (8) What are the inhomogeneous Maxwell's equations (in covariant form) that result from the Lagrangian density in (a)?

   c. (8) If photons were to have a rest mass proportional to some parameter μ, how should the Lagrangian from (a) be modified?

d. (8) If photons have rest mass, what is the mathematical form of the electrostatic potential of an isolated point charge q?

5. The symmetrical stress tensor is $\Theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\lambda}F^{\mu\lambda}\right).$

   a. (8) What is the physical interpretation of the equation, $\partial_\alpha\Theta^{\alpha\beta} = 0$, for $\beta = 0$?

   b. (8) What is the physical interpretation of the equation, $\partial_\alpha\Theta^{\alpha\beta} + \frac{1}{c}F^{\beta\lambda}J_\lambda = 0$, for $\beta = i$?

6. A current density produces EM fields satisfying the covariant equations, $\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta$. Suppose you solve it in the Lorenz gauge, where $\partial_\alpha A^\alpha = 0$.

   a. (8) Changing the RHS to $\delta^{(4)}(x - x')$, what is the 4D retarded Green's function $G_r(x - x')$ for this equation?

   b. (8) How do write the most general solution for $A^\alpha(x)$ using the retarded Green's function?

7. (8) Describe the physical meaning of the formula below and the definitions of $U(\tau)$ and τ_0.

$$A^\alpha(x) = \left[\frac{eU^\alpha(\tau)}{U \cdot (x - r(\tau))}\right]_{\tau=\tau_0}$$

8. (8) Write a formula that is dimensionally correct (up to a numerical constant) for the instanteous power radiated by an accelerated charge in *nonrelativistic motion.*

9. (8) Generalize your result of the previous question to a covariant formula for an arbitrary relativistic case.

10. (8) The angular distribution of radiated power from an accelerated charge is defined from the formula

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\mathbf{n} \times \left[(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^5}.$$

Physically, how does the angular distribution of power radiated change as one goes from the nonrelativistic case to the extreme relativistic limit?

Electrodynamics II. Exam 3. Part B (100 pts.)
Relativistic Particles, Fields and Radiation Open Book

Name______________________________ KSU 2015/05/11 14:00 - 15:50

Instructions: Please show the details of your derivations. Explain your reasoning for full credit. Open-book and 1-page note summary allowed.

1. A point charge e with mass m moves on a trajectory with 4-vector $r(\tau)$ and 4-velocity $U(\tau)$ parametrized by proper time τ.

   a. (10) Write the covariant equation of motion for the 4-velocity of the charge in an electromagnetic field described by EM field tensor $F^{\alpha\beta}$.

   b. (15) Write out separately the equation of motion for the time component, $dU^0/d\tau$, showing how $F^{\alpha\beta}$ becomes the EM fields $\mathbf{E}$ and $\mathbf{B}$ in the equation.

   c. (15) Write out separately the equations of motion for space components, $dU^i/d\tau$, showing how $F^{\alpha\beta}$ becomes the EM fields $\mathbf{E}$ and $\mathbf{B}$ in the equations.

2. A particle of charge q, mass m, and initial speed v_0 is incident on a path near a fixed charge Q, at an impact parameter b large enough so that the particle's path remains nearly a straight line as it passes Q.

   a. (20) Assuming nonrelativistic motion, get the instantaneous acceleration and then write an expression for the instantaneous power radiated as a function of the separation r between q and Q.

   b. (10) If the particle is now assumed to move at constant speed on a straight line (it is a contradiction to part (a) but gives a way to complete the calculation), write the relation for $r(t)$.

   c. (20) Combining the above results, write an integration over dr to determine the total energy ΔW radiated as the particle with (q, m) passes the fixed charge Q.

   d. (10) Evaluate the integral from part c to get the net energy radiated.

3. Consider the relativistic Liénard formula for radiation from an accelerated charge with instantaneous momentum $\mathbf{p} = \gamma mc\vec{\beta}$:

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right].$$

a) (12) Show that the above equation can also be written as

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^4\left[(\dot{\vec{\beta}})^2 + \gamma^2(\vec{\beta}\times\dot{\vec{\beta}})^2\right].$$

b) (16) From the relativisitic version of Newton's Second Law show that

$$\dot{\vec{\beta}} = \frac{1}{mc\gamma}\left[\vec{F} - \left(\vec{\beta}\cdot\vec{F}\right)\vec{\beta}\right]$$

c) (12) Combine results a and b to get the instantaneous radiated power in terms of $\vec{F}$.