Electrodynamics II Exam 2. Part A (130 pts.) Closed Book Scattering, Special Relativity

Name G. WYSIN KSU 2015/04/13 8:00 a.m.

You may use SI units for scattering questions and CGS units for relativity topics. No derivations here, just state your responses clearly, and define your variables in words.

1. (8) In scattering problems, the main goal is to find the differential scattering cross section, $d\sigma/d\Omega$. How do you calculate this? Give a formula and explain.

$$\frac{d\sigma}{d\Omega} = \frac{dP_{scatt}/d\Omega}{|\vec{S}_{inc}|} \quad \text{where} \quad \frac{dP_{scatt}}{d\Omega} = \frac{1}{2}\mathrm{Re}\left\{r^2\hat{n}\cdot(\vec{E}\times\vec{H}^*)_{scatt}\right\}$$

$$\text{and} \quad |\vec{S}_{inc}| = \frac{1}{2}\mathrm{Re}\left\{\vec{E}\times\vec{H}^*\right\}_{\text{incident waves}}$$

2. (6) What is the definition of scattering in the Rayleigh limit?

$a \ll \lambda$ where a = size or radius of scatterer.
λ = wavelength of light scattered.

3. (6) State physically why it helps to use the Rayleigh limit.

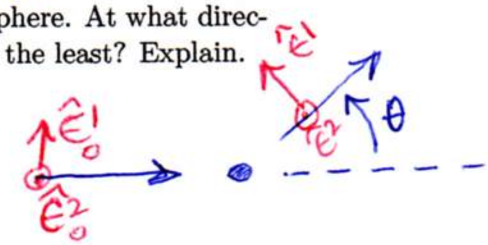
If $a \ll \lambda$ the electric and magnetic fields will be almost uniform within the volume of a scatterer.

4. (8) In the Rayleigh limit, how does the total scattering cross section of unpolarized light from a small dielectric sphere depend on the wavelength and the radius of the sphere?

$\sigma \propto k^4 V^2 \propto \frac{1}{\lambda^4} a^6$. where $k = \frac{2\pi}{\lambda}$ = wavevector.
$V = \frac{4\pi}{3}a^3$ is volume

5. (8) Unpolarized monochromatic light scatters from a small dielectric sphere. At what direction relative to the incident beam will the total scattered intensity be the least? Explain.

$\hat{\epsilon}^1$ polarization has $\frac{d\sigma_{\parallel}}{d\Omega} \sim \cos^2\theta$.
$\hat{\epsilon}^2$ polarization has $\frac{d\sigma_{\perp}}{d\Omega} \sim 1$



Total $\sim 1+\cos^2\theta$ has minimum at $\theta = \pi/2$.

6. (8) Monochromatic light polarized perpendicular to the scattering plane (defined by $\vec{k}$ and $\vec{k}'$) scatters from a small dielectric sphere. At what direction relative to the incident beam will the total scattered intensity be the least? Explain.

Incident light only has $\hat{\epsilon}_0^2$. This induces elec. dipole only $\perp$ to scattering plane. There will be no scattered radiation $\perp$ to the scattering plane. mins. at $\theta = \pi/2$, $\phi = \pm\pi/2$.

7. (8) Give a statement of the Postulate of Relativity (as assumed by Einstein, Poincaré, etc.).

The Laws of Physics and the results of experiments are the same for all inertial reference frames.

8. (8) Give a statement of the other Postulate that Albert Einstein used in the development of the Special Theory of Relativity.

The speed of light is the same finite value for all observers,

or: There is a finite limiting speed c for all objects.

9. (8) Two events viewed in an inertial reference frame K occur at space-time points $(ct_a, \vec{x}_a)$ and $(ct_b, \vec{x}_b)$. If the *same two events* are viewed in another inertial frame K' in motion relative to K, what quantity is invariant in the two frames?

The squared space-time interval is invariant.

$(ct_a - ct_b)^2 - |\vec{x}_a - \vec{x}_b|^2 =$ invariant in all frames.

10. (8) For a Lorentz boost along the x^1 direction, where the K' frame moves at speed $v = \beta c$ relative to K, write out the 4×4 transformation matrix that gives K coordinates (x^0, x^1, x^2, x^3) in terms of K' coordinates (x'^0, x'^1, x'^2, x'^3) (using the contravariant coordinates here and rapidity ζ). origin of K' has position $x = vt$ in K.

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$$

$\cosh\zeta = \gamma$

$\sinh\zeta = \beta\gamma$

$\tanh\zeta = \beta$

11. (8) A particle (or a clock) is moving with some varying velocity $\beta(t)$ as viewed in a lab frame K from an initial time t_a to final time t_b. How do you write an expression for the time interval in the reference frame of the particle (or, the time elapsed on the moving clock).

The clock is at rest in its inertial frame, so $d\vec{x}' = 0$. Use invariant interval. $ds^2 = c^2 dt^2 - d\vec{x}^2 = c^2 d\tau^2$ $\quad d\tau = dt' =$ proper time

Finite value $\Delta\tau = \int d\tau = \int dt \sqrt{1 - \frac{d\vec{x}^2}{c^2 dt^2}} = \int_{t_a}^{t_b} dt \sqrt{1 - \vec{\beta}^2(t)}$.

$\vec{\beta}(t) = \frac{d\vec{x}}{dt}$.

12. (6) The energy-momentum 4-vector is $p^\alpha = (E/c, \vec{p})$. What does its invariant squared length depend on?

$$p_\alpha p^\alpha = (E/c)^2 - \vec{p}^{\,2} = m^2c^2$$ i.e. the rest mass squared.

13. (8) The contravariant components of space-time 4-vector $x^\alpha = (ct, x^1, x^2, x^3)$ in frame K undergo a Lorentz transformation to a moving (K') frame by $x'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} x^\beta$. Write an equation that expresses how the 4-momentum p^α is transformed to the K' frame.

All 4-vectors transform the same way as x^α.

$$p'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} p^\beta$$

14. (8) Consider an infinitesimal Lorentz boost along the x^1 direction with rapidity $\zeta \ll 1$. Write out the matrix K_1 that is the generator of these boosts.

From the inverse of the matrix in #10, with $\zeta \ll 1$, we get

$$x' = Lx$$

$$L = \begin{pmatrix} 1 & -\zeta & 0 & 0 \\ -\zeta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\zeta, \quad K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

15. (8) If you have the three generators of boosts along the Cartesian axes, K_1, K_2, K_3, how do you write the formal operator $A_{\text{boost}}(\vec{\beta})$ that produces a pure Lorentz boost with velocity $\vec{v} = \vec{\beta}c$?

$$A_{\text{boost}}(\vec{\beta}) = e^{-\vec{\zeta}\cdot\vec{K}}$$ where $\vec{\zeta} = \hat{\beta}\tanh^{-1}\beta$.

16. (8) Write a formal definition of the components of the electromagnetic field tensor $F^{\alpha\beta}$ in terms of the 4-potential A^α.

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha.$$

17. (8) Write the equation for electromagnetic or Lorentz "4-force" on a charge q in covariant form. Hint: You need to use $F^{\alpha\beta}$ and produce a force-like quantity that transforms as a 4-vector.

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta$$ where $U_\beta = \frac{1}{m} p_\beta$ is the 4-velocity.

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Please show the details of your derivations here. Explain your reasoning for full credit. Open-book and 2-page note summary allowed.

1. The threshold kinetic energy T_{th} in the laboratory for a given reaction is the kinetic energy of the incident particle on a stationary target just sufficient to make the center of mass energy W equal to the sum of the rest energies in the final state.

   Consider a pi-meson photoproduction reaction (photon γ incident on a stationary proton),

$$\gamma\, p \longrightarrow \pi^0\, p.$$

   The rest energies are $m_p = 938.5$ MeV for the proton and $m_\pi = 135.0$ Mev for the pion ($c = 1$ units).

   a) (10) What minmum total energy W must be present in the "center of mass" frame for this reaction to be possible? (This shouldn't require much calculation.)

   b) (25) Use conservation of 4-momentum and invariant scalar products to determine the threshold photon energy E_γ needed for this reaction, in the lab frame, in MeV.

   c) (25) Using a Lorentz transformation (or other method), find the velocity β of the center of mass frame when the photon has the threshold energy.

   d) (Bonus, 20) How large is the final state proton energy, if this "collision" could be considered as head-on, with momentum components only along the photon direction?

a) In c.o.m. frame, total energy = rest mass = $m_\pi + m_p$.

$$W = m_\pi + m_p = (135 + 938.5)\text{ MeV} = 1074 \text{ MeV}.$$

b) Let p_1, p_2 be lab frame momenta of π and proton.
Let p_3, p_4 be those transformed to c.o.m. frame, where $\vec{p}_3 = -\vec{p}_4$ (all measured before the reaction). There is zero total momentum in the c.o.m frame.

$(p_1 + p_2)^2 = (p_3 + p_4)^2$. Note: $p_3 + p_4 = (W, 0)$

$p_1^2 + p_2^2 + 2 p_1 \cdot p_2 = W^2$

$m_\gamma^2 + m_p^2 + 2(E_\gamma E_p - \vec{p}_\gamma \cdot \vec{p}_p) = W^2$ but $m_\gamma^2 = 0$. $\vec{p}_p = 0$.

Then $W^2 = m_p^2 + 2 E_\gamma m_p$.

b) Combine with result of a).

$$m_p^2 + 2E_\gamma m_p = (m_p + m_\pi)^2 = m_p^2 + m_\pi^2 + 2m_p m_\pi$$

$$2E_\gamma m_p = m_\pi^2 + 2m_p m_\pi .$$

$$\boxed{E_\gamma = m_\pi + \frac{m_\pi^2}{2m_p}} \rightarrow 135 + \frac{135^2}{2(938.5)} = 144.7 \text{ MeV}$$

c) velocity of c.o.m. frame?

$\xrightarrow{\gamma} \quad \xleftarrow{P}$

Transform via Lorentz transformation.

equal & opposite momenta. viewed in c.o.m.

$$p_\gamma' = \gamma(p_\gamma - \beta E_\gamma), \quad p_p' = \gamma(0 - \beta m_p)$$

These are equal and opposite in c.o.m. frame.

$$\gamma(p_\gamma - \beta E_\gamma) = -(-\gamma\beta m_p)$$

$$p_\gamma = \beta(m_p + E_\gamma)$$

$$\boxed{\beta = \frac{p_\gamma}{m_p + E_\gamma}} = \frac{E_\gamma}{m_p + E_\gamma} = \frac{144.7}{938.5 + 144.7} = 0.1336$$

d) Bonus. Final state proton energy.

In a head-on collision, the particles just reverse their c.o.m. momenta.

Final proton momentum in c.o.m. is $p_p' = +\gamma\beta m_p$, and $E_p' = \gamma m_p$.

Transform back to lab frame,

$$E_p = \gamma(E_p' + \beta p_p') = \gamma(\gamma m_p + \beta \cdot \gamma\beta m_p) = \gamma^2(1+\beta^2)m_p$$

proton

$$\boxed{E_p = \left(\frac{1+\beta^2}{1-\beta^2}\right)m_p} = \left(\frac{1+0.1336^2}{1-0.1336^2}\right)938.5 = 972.6 \text{ MeV}$$

Final proton energy

Also for the photon: $p_\gamma' = -\gamma\beta m_p, \quad E_\gamma' = \gamma\beta m_p$.

Then $E_\gamma = \gamma(E_\gamma' + \beta p_\gamma') = \gamma(\gamma\beta m_p - \beta\gamma\beta m_p) = \gamma^2\beta(1-\beta)m_p \rightarrow 110.6 \text{ MeV}$

2. In an inertial reference frame K there are static and uniform electric field E_0 parallel to the x-axis and magnetic induction $B_0 = 2E_0$ in the xy-plane at angle θ to the x-axis.

   a) (25) Make a Lorentz boost of these fields with velocity $\vec{\beta} = \beta\hat{z}$ along the z-axis, via Jackson Eq. (11.149):

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

   Give the results for all field components in the K' frame.

   b) (25) Look for a reference frame K' where $\vec{E}'$ and $\vec{B}'$ are parallel. Determine the needed boost velocity $\vec{\beta}$, by forcing $\vec{E}'\times\vec{B}'$ to be zero. You should obtain β as a function of θ in the original frame.

   c) (10) In the limit $\theta = \pi/2$, what boost speed β is needed to make $\vec{E}'$ and $\vec{B}'$ parallel?

a) Fields $\vec{E}, \vec{B}$ are $\perp$ to $\hat{z}$ or $\vec{\beta}$. So $\vec{\beta}\cdot\vec{E} = 0$, $\vec{\beta}\cdot\vec{B} = 0$.

$$E_1' = \gamma(E_1 - \beta B_2) \qquad B_1' = \gamma(B_1 + \beta E_2)$$

$$E_2' = \gamma(E_2 + \beta B_1) \qquad B_2' = \gamma(B_2 - \beta E_1)$$

where $1 = x$, $2 = y$ components.

b) To make $\vec{E}' \parallel \vec{B}'$, the cross product $\vec{E}'\times\vec{B}' = 0$ has only a z-component.

$$(\vec{E}'\times\vec{B}')_3 = E_1'B_2' - E_2'B_1'$$

$$= \gamma^2(E_1 - \beta B_2)(B_2 - \beta E_1) - \gamma^2(\overset{0}{\underset{\|}{E_2}} + \beta B_1)(B_1 + \beta \overset{0}{\underset{\|}{E_2}}) = 0$$

$$E_1 B_2(1+\beta^2) - \beta(E_1^2 + B_2^2) - \beta B_1^2 = 0$$

$$E_0 \overbrace{2E_0 \sin\theta}(1+\beta^2) - \beta(4\sin^2\theta + 1)E_0^2 - \beta\cdot 4E_0^2\cos^2\theta = 0$$

$$2\sin\theta(1+\beta^2) - \beta(1 + 4\sin^2\theta + 4\cos^2\theta) = 0$$

$$2\sin\theta\cdot\beta^2 - 5\beta + 2\sin\theta = 0$$

... Solve for β.

$$\beta = \frac{1}{4\sin\theta}\left[5 \pm \sqrt{25 - 4(2\sin\theta)^2}\,\right] = \frac{5}{4\sin\theta}\left[1 - \sqrt{1 - \left(\tfrac{4}{5}\sin\theta\right)^2}\,\right]$$

need to choose $\ominus$ root so $\beta \to 0$ when $\theta \to 0$.

Thus boost along $\hat{z}$ will bring $\vec{E}'$ and $\vec{B}'$ to be parallel.

c) As $\theta \to \pi/2$ we have $\sin\theta \to 1$.

$$\beta = \frac{5}{4}\left[1 - \sqrt{1 - \left(\tfrac{4}{5}\right)^2}\,\right] = \frac{5}{4}\left[1 - \sqrt{\tfrac{9}{25}}\,\right] = \frac{5}{4}\left(1 - \frac{3}{5}\right) = \frac{5}{4}\cdot\frac{2}{5}$$

$\beta = 1/2$.