Electrodynamics II Exam 3. Part A (140 pts.) Closed Book Radiation from Acceleration

Name G. Wysin KSU 2006/05/09 14:00 - 15:50

Instructions: Some small derivations here, state your responses clearly, define your variables in words if they are not common usage. Electron constants: $r_0 = \frac{e^2}{mc^2} = 2.82 \times 10^{-15}$m. $\tau = \frac{2}{3}\frac{e^2}{mc^3} = 6.26 \times 10^{-24}$s.

1. A point charge q moves on the trajectory $r^\alpha(\tau)$ with 4-velocity $U^\alpha(\tau)$, where τ is the proper time.

   a) (8) Write a formula that gives its 4-current $J^\alpha(x)$ at space-time point x, which is the source current in the wave equation for the 4-potential, under the Lorentz gauge,

$$\partial_\beta \partial^\beta A^\alpha(x) = \frac{4\pi}{c} J^\alpha(x)$$

As the ρ and $\vec{J}$ are

$$\left.\begin{array}{l}\rho(\vec{x}) = e\,\delta(\vec{x}-\vec{r}(t)) \\ \vec{J}(\vec{x}) = e\vec{v}\,\delta(\vec{x}-\vec{r}(t))\end{array}\right\} \text{combine to} \quad \begin{array}{l} J^0 = c\rho = ec\,\delta(\vec{x}-\vec{r}) \\ \vec{J} = J^i = ev_i\,\delta(\vec{x}-\vec{r})\end{array}$$

or better $\boxed{J^\alpha(x) = ec\int d\tau\, U^\alpha(\tau)\,\delta^4(x-r(\tau))}$

   b) (8) Write a covariant expression for the retarded Green function $G_r(x-x')$ needed to solve this wave equation.

$$G_r(x-x') = \frac{1}{2\pi}\Theta(x_0-x_0')\,\delta[(x-x')^2]$$

arguments are 4-vectors, and $(x-x')^2 = c^2(t-t')^2 - |\vec{x}-\vec{x}'|^2$.

   c) (12) Show how to use the retarded Green function to get the solution for $A^\alpha(x)$ expressed in covariant form (i.e., the Liénard-Wiechert potentials).

$$A^\alpha(x) = \int d^4x'\, G_r(x-x')\,\frac{4\pi}{c} J^\alpha(x')$$

$$= \frac{4\pi}{c}\int d^4x'\, \frac{\theta(x_0-x_0')}{2\pi}\,\delta[(x-x')^2]\; ec\int d\tau\, U^\alpha(\tau)\,\delta^4(x'-r(\tau))$$

$$= 2e\int d\tau\, \theta(x_0-x_0')\,U^\alpha(\tau)\,\delta[(x-r(\tau))^2]$$

$$= 2e\int d\tau\, \theta(x-0)\,U^\alpha(\tau)\,\frac{\delta(\tau-\tau_0)}{2(x-r(\tau_0))\cdot\frac{dr}{d\tau}\big|_{\tau_0}}$$

$$= \frac{e\,U^\alpha(\tau_0)}{(x-r(\tau_0))\cdot U(\tau_0)}$$ where τ_0 solves $(x-r(\tau_0))^2 = 0$ with $ct > r_0(\tau_0)$.

2. (8) If a charge is moving on some path *non-relativistically* with a given velocity $\vec{v}(t) = c\vec{\beta}(t)$, what is the formula for the total instantaneous radiated power?

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{dt}\right)^2 = \frac{2}{3}\frac{e^2}{c^3}\dot{\vec{v}}^2$$

3. (8) If a charge is moving on some path *relativistically* with a given 4-momentum $p(\tau)$, what is the covariant formula for the total instantaneous radiated power?

$$P = -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{dp}{d\tau}\cdot\frac{dp}{d\tau} = -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{dp_\alpha}{d\tau}\frac{dp^\alpha}{d\tau}$$

4. (8) For a charge in arbitrary relativisitic motion, what is a formula for the part of its electric field that produces radiated power?

$$\vec{E}_{rad} = \frac{e}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\hat{n}\cdot\vec{\beta})^3 R}\right]_{ret} \quad \text{where } R = |\vec{x}-\vec{x}'|$$

5. (12) Generally, explain what happens to the angular distribution of radiated power from an accelerated charge when the motion is strongly relativistic (compared to the non-relativistic case).

The factor $(1-\hat{n}\cdot\vec{\beta})$ in denominator of $\vec{E}$ causes the radiation to get concentrated more towards the "forward direction", i.e., near the direction of $\vec{\beta} = \vec{v}/c$.

6. (8) If an energtic charge undergoes relativistic cyclotron motion at frequency ω_0, what range of frequencies will typicallly be present in the spectrum of its radiation?

The concentration of the emission in the forward direction produces frequencies typically from ω_0 up to $\gamma^3\omega_0$.

Recall the Liénard result for the instantaneous power radiated by an accelerated charge is

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^6[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2]$$

7. An electron of total energy 5.11 MeV is undergoing linear acceleration due to application of a uniform electric field of strength $E_0 = 1.0$ MV/m. $(mc^2 = 0.511\,\text{MeV})$

a) (12) Rewrite the Liénard result to express the radiated power for this case, in terms of the applied force.

$\vec{\beta}\times\dot{\vec{\beta}} = 0$ since $\vec{\beta}$ and $\dot{\vec{\beta}}$ are parallel. Or, just realize,

P depends on $\frac{dp}{dt} = \frac{d}{dt}(\gamma m v) = \frac{d}{dt}\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) = \frac{d}{dt}\left(\frac{mc}{\sqrt{\frac{c^2}{v^2}-1}}\right)$

$$F = \frac{dp}{dt} = \frac{mc}{\left(\frac{c^2}{v^2}-1\right)^{3/2}}\frac{c^2}{v^3}\dot{v} = \frac{m\dot{v}}{(1-v^2/c^2)^{3/2}} = \gamma^3 mc\dot{\beta}$$

$$P = \frac{2}{3}\frac{e^2}{c}\frac{1}{(mc)^2}\left(\gamma^3 mc\dot{\beta}\right)^2 = \frac{2}{3}\frac{e^2}{m^2c^3}F^2$$

b) (6) How large is the force, in newtons?

$$F = eE_0 = (1.602\times10^{-19}\,\text{C})(10^6\,\tfrac{\text{N}}{\text{C}}) = 1.6\times10^{-13}\,\text{N}$$

c) (6) Calculate the instantaneous radiated power in watts.

$$P = \frac{2}{3}\left(\frac{e^2}{mc^2}\right)\left(\frac{1}{mc}\right)F^2 = \frac{2}{3}(2.82\times10^{-15}\,\text{m})\frac{(1.6\times10^{-13}\,\text{N})^2}{(9.11\times10^{-31}\,\text{kg})(2.998\times10^{8}\,\text{m/s})}$$

$$P = 1.76\times10^{-19}\ \text{watts.}$$

8. (8) Analysis of conservation of energy in a radiation problem leads to what result for the radiation reaction force F_{rad}?

$\vec{F}_{\text{rad}} = m\tau\ddot{\vec{v}}$ where $\tau = \frac{2}{3}\frac{e^2}{mc^3}$

9. An electron of total energy 5.11 MeV is undergoing cyclotron motion in a uniform magnetic induction of strength $B_0 = 3.33$ mT.

a) (12) Rewrite the Liénard result (page 3) to express the radiated power for this case, in terms of the applied force.

Now $(\vec{\beta} \times \dot{\vec{\beta}})^2 = (\beta\dot{\beta})^2$ since these are perpendicular.

$(\dot{\vec{\beta}})^2 = \dot{\beta}^2$ will be related to the force, $F = m\dot{\beta}c\gamma$

In this case, γ does not change with time.

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[\dot{\beta}^2 - \beta^2\dot{\beta}^2\right] = \frac{2}{3}\frac{e^2}{c}\gamma^4\dot{\beta}^2 = \frac{2}{3}\frac{e^2}{c}\frac{1}{(mc)^2}\gamma^2(m\dot{\beta}c\gamma)^2$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2 F^2,$$ larger than that for linear acceleration, by the factor γ^2.

b) (6) How large is the force, in newtons?

(SI) $\vec{F} = e\vec{v} \times \vec{B}$, where $\gamma = \frac{E}{E_0} = \frac{5.11\,\text{MeV}}{0.511\,\text{MeV}} = 10 = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = 0.995$

$$F = e\beta c B_0 = (1.6\times10^{-19}\,\text{C})(0.995)(2.998\times10^{8}\,\tfrac{\text{m}}{\text{s}})(3.33\times10^{-3}\,\text{T}) = 1.59\times10^{-13}\,\text{N}$$

c) (6) Calculate the instantaneous radiated power in watts.

$$P = \frac{2}{3}\left(\frac{e^2}{mc^3}\right)\left(\frac{1}{mc}\right)\gamma^2F^2 = \frac{2}{3}(2.82\times10^{-15}\,\text{m})\frac{10^2\times(1.59\times10^{-13}\,\text{N})^2}{(9.11\times10^{-31}\,\text{kg})(2.998\times10^{8}\,\text{m/s})}$$

$$P = 1.74\times10^{-17}\ \text{watts.}$$

10. (12) The characteristic time below which radiation reaction forces are important is $\tau = \frac{2}{3}\frac{e^2}{mc^3}$. For an electron undergoing cylcotron motion in a uniform magnetic induction B, at approximately what strength of B (in tesla) do the reaction forces become very important to the problem?

The reaction force is very important when the cyclotron period becomes as short as τ. The angular frequency is $\omega = \frac{eB}{m\gamma}$

reaction important $\Rightarrow \omega\tau \gg 1$ or $\frac{eB}{m\gamma}\tau \gg 1$, Assume $\gamma = 10$

$$B \gg \frac{m\gamma}{e\tau} = \frac{(9.11\times10^{-31}\,\text{kg})\,(10)}{(1.6\times10^{-19}\,\text{C})(6.26\times10^{-24}\,\text{s})} = 9.1\times10^{12}\ \text{tesla.}$$

(All SI units here)

Electrodynamics II Exam 3. Part B (72 pts.) Open Book Radiation from Acceleration

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Instructions: Please show the details of your derivations. Explain your reasoning for full credit. Open-book and 1-page note summary allowed.

1. A particle of charge q and mass m is accelerated linearly from rest by a constant force F applied during a short time interval, $0 < t < t_0$, after which the force is turned off. We are interested in how to calculate the radiation properties. The motion is relativistic.

   a) (12) Calculate exactly the total energy radiated due to this acceleration.

   b) (16) For the rest of this question, we need to know $\beta(t)$ and $\dot{\beta}(t)$. Find exact expressions for the time dependence of these in terms of F and other needed constants.

   c) (16) Write an expression (an integral over time) that will give the angular distribution of the radiated energy, $\frac{dE}{d\Omega}$. You do not need to evaluate it, but make sure it shows the dependence on angular coordinates out to a distant observer.

   d) (12) Determine the total distance travelled by the particle over the interval $0 < t < t_0$.

   e) (16) Consider the radiated energy you obtained in part a), and the distance you obtained in d). If the motion was highly relativistic, how large was the average radiation reaction force acting on the charge? Under what conditions will the reaction effects crucially change the motion in this problem?

a) $P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{dt}\right)^2$ For linear acceleration. But $\frac{d\vec{p}}{dt} = \vec{F}$

There is a constant force, so there is a constant power until $t=t_0$.

Energy radiated:

$$E_{rad} = \int_0^{t_0} P\,dt = \int_0^{t_0} \frac{2}{3}\frac{e^2}{m^2c^3}F^2\,dt = \underline{\frac{2}{3}\frac{e^2}{m^2c^3}F^2 t_0.}$$

b) As $\frac{dp}{dt} = F = \text{constant}$, get $\boxed{p(t) = Ft} = \frac{mv}{\sqrt{1-v^2/c^2}} = \frac{m\beta c}{\sqrt{1-\beta^2}}$

Squaring, $(Ft)^2(1-\beta^2) = (mc)^2\beta^2$, $(Ft)^2 = ((Ft)^2+(mc)^2)\beta^2$

$$\boxed{\beta(t) = \frac{Ft}{\sqrt{m^2c^2+F^2t^2}}}$$

Also could get this by writing,

$$\beta(t) = \frac{v}{c} = \frac{\gamma m v}{\gamma m c} = \frac{p}{E/c} = \frac{p}{\sqrt{m^2c^2+p^2}}$$

b)... For linear acceleration, $\frac{dp}{dt} = F = \frac{d}{dt}(\gamma m v) = \gamma^3 m\dot{v} = \gamma^3 mc\dot{\beta}$.

$$\boxed{\dot{\beta}(t) = \frac{F}{\gamma^3 mc} = \frac{F}{\left(\sqrt{m^2c^2+F^2t^2}/mc\right)^3 mc} = \frac{Fm^2c^2}{(m^2c^2+F^2t^2)^{3/2}}}$$

$$\gamma = \frac{E}{mc^2} = \frac{\sqrt{m^2c^4+p^2c^2}}{mc^2} = \frac{\sqrt{m^2c^2+F^2t^2}}{mc}, \quad \gamma mc = \sqrt{m^2c^2+F^2t^2}$$

Better: $$\boxed{\dot{\beta}(t) = \frac{F/mc}{(1+F^2t^2/m^2c^2)^{3/2}}}$$

c) $$\frac{dE}{d\Omega} = \int dt \frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c}\int dt \frac{\left|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right|^2}{(1-\hat{n}\cdot\vec{\beta})^5} \quad \text{with } \vec{\beta}\times\dot{\vec{\beta}} = 0$$

Then simplify top, $|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})|^2 = |\hat{n}\times\dot{\vec{\beta}}|^2 = \dot{\beta}^2\sin^2\theta$

where θ is the angle between $\hat{n}$ and the force direction.

Also have $\hat{n}\cdot\vec{\beta} = \beta\cos\theta$.

$$\frac{dE}{d\Omega} = \frac{e^2}{4\pi c}\int_0^{t_0} dt \frac{(F/mc)^2\sin^2\theta}{(1+F^2t^2/m^2c^2)^3}\cdot\frac{1}{\left[1-\frac{Ft/mc}{\sqrt{1+F^2t^2/m^2c^2}}\cdot\cos\theta\right]^5}$$

d) Distance travelled. $\Delta x = \int_0^{t_0}\beta c\,dt = \int_0^{t_0}\frac{cFt\,dt}{\sqrt{m^2c^2+F^2t^2}} = \frac{c}{F}\sqrt{m^2c^2+F^2t^2}\Big|_{t=0}^{t=t_0}$

$$\boxed{\Delta x = \frac{c}{F}\left[\sqrt{m^2c^2+F^2t_0^2} - mc\right]} = \frac{\Delta K}{F} = \frac{\text{change in kinetic energy}}{\text{force}}.$$

e) If highly relativistic, then $Ft_0 \gg mc$, and $\Delta x \simeq ct_0$ ($\langle\beta\rangle \simeq 1$).

Get the reaction force (averged value) by dividing radiated energy by Δx.

$$\langle F_{rad}\rangle = \frac{E_{rad}}{\Delta x} = \frac{2}{3}\frac{e^2}{m^2c^3}F^2t_0\frac{1}{ct_0} = \frac{2}{3}\frac{e^2}{m^2c^4}F^2$$

This is very significant if $\langle F_{rad}\rangle \gtrsim F$.

or $\frac{2}{3}\frac{e^2}{m^2c^4}F \gtrsim 1. \implies \boxed{F \gtrsim \frac{3}{2}\frac{m^2c^4}{e^2} = \frac{mc^2}{r_0}}$

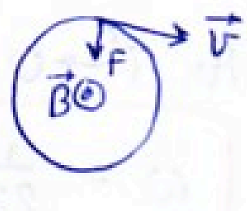
2. (BONUS) This problem considers radiation reaction effects on a non-relativistic electron performing cylcotron motion (xy-plane) in a uniform magnetic induction $\vec{B} = B\hat{z}$. Assume the radiation reaction force is $F_{\text{rad}} = m\tau\ddot{\vec{v}}$, where $\tau = \frac{2}{3}\frac{e^2}{mc^3}$.

a) (12) Write the differential equations of motion for the components of the velocity, $v_x(t)$ and $v_y(t)$, including the Lorentz and radiation reaction forces.

b) (16) Partially solve the equations, assuming the usual harmonic dependence like $e^{-i\omega t}$. Determine ω, but not any constants of integration. Note that this complex ω is different than the unperturbed cyclotron frequency, $\omega_B = \frac{eB}{mc}$.

c) (12) When radiation reaction effects are weak, determine how the velocity components decay with time.

a) I'll take the charge as q, could be ⊕ or ⊖.



The motion is nonrelativistic. The net force is the combination of $\vec{F}_{\text{lorentz}} = q\frac{\vec{v}}{c}\times\vec{B}$ and $\vec{F}_{\text{rad}} = m\tau\ddot{\vec{v}}$

$$q\frac{\vec{v}}{c}\times\vec{B} + m\tau\ddot{\vec{v}} = m\dot{\vec{v}}$$

let $\vec{v} = (v_x, v_y)$
$\vec{B} = B\hat{z}$.

Split equation into components.
Also useful to re-arrange it.

$$\boxed{\begin{aligned} m(\dot{v}_x - \tau\ddot{v}_x) &= \frac{q}{c}B v_y \\ m(\dot{v}_y - \tau\ddot{v}_y) &= -\frac{q}{c}B v_x . \end{aligned}}$$

b) Assuming $\vec{v}(t) = \vec{v}_0 e^{-i\omega t}$, then $\frac{d}{dt} \to -i\omega$, and get,

$$-i\omega v_x - \tau(-i\omega)^2 v_x = \frac{qB}{mc} v_y$$

Define $\frac{qB}{mc} \equiv \omega_B$

$$-i\omega v_y - \tau(-i\omega)^2 v_y = -\frac{qB}{mc} v_x$$

rewrite as

$$-i\omega(1+i\omega\tau)v_x = \omega_B v_y$$

$$-i\omega(1+i\omega\tau)v_y = \omega_B v_x$$

eliminate v_y to get an eqn. just for the frequency ω.

$$-i\omega(1+i\omega\tau)\, v_x = \omega_B \cdot \frac{-\omega_B}{-i\omega(1+i\omega\tau)}\, v_x$$

$$\omega^2(1+i\omega\tau)^2 = \omega_B^2 .$$ It's actually 4th order, but do sqrt.

$$\omega(1+i\omega\tau) = \omega_B$$ Now solve the quadratic. Maybe RHS needs $\pm$

$$i\tau\omega^2 + \omega - \omega_B = 0$$

$$\omega = \frac{1}{2i\tau}\left[-1 \pm \sqrt{1+4i\tau\omega_B}\right].$$

As $\tau \to 0$ we must recover $\omega = \omega_B$, so use $+$ root.

$$\boxed{\omega = \frac{1}{2i\tau}\left[-1 + \sqrt{1+4i\tau\omega_B}\right].}$$

c) Weak radiation effects correspond to $\tau\omega_B \ll 1$. Then expand,

$$\omega \simeq \frac{1}{2i\tau}\left[-1 + \left(1 + \tfrac{1}{2}(4i\tau\omega_B) - \tfrac{1}{8}(4i\tau\omega_B)^2\right)\right]$$

$$\omega \approx \frac{1}{2i\tau}\left[\cancel{-1} + \cancel{1} + 2i\tau\omega_B + 2\tau^2\omega_B^2\right]$$

$$\boxed{\omega = \omega_B - i\tau\omega_B^2}$$

Then the time-dependences of velocity components will be as

$$e^{-i\omega t} = e^{-i\omega_B t}\, e^{-i(-i\omega_B^2\tau)t}$$

$$= e^{-i\omega_B t}\, e^{-\omega_B^2\tau t}.$$

Exp. Decay over time scale of $\frac{1}{\omega_B^2\tau}$.