Electrodynamics II Exam 1. Part A (176 pts.) Closed Book Radiation and Scattering

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Instructions: Use SI units. No derivations here, just state your responses clearly, and define your variables in words.

1. (12) What are the general solutions for the vector and scalar potentials $\vec{A}(\vec{r},t)$ and $\Phi(\vec{r},t)$ when the sources $\vec{J}(\vec{r},t)$ and $\rho(\vec{r},t)$ have harmonic time dependence?

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \vec{J}(\vec{x}')\cdot e^{-i\omega t}$$

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \rho(\vec{x}')\cdot e^{-i\omega t}$$

2. (12) In radiation problems, explain why you can usually obtain $\vec{E}$ and $\vec{H}$ by using only the vector potential $\vec{A}$. If you know $\vec{H}$, how do you get $\vec{E}$. Where does this really apply?

In a region where $\rho=0$ and $\vec{J}=0$, i.e. outside the source, you can always apply Maxwell-Amp. Law to relate $\vec{E}$ and $\vec{H}$. In harmonic form,

$$\vec{\nabla}\times\vec{H} = -i\omega\vec{D} \Rightarrow \vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{i}{\epsilon_0\omega}\vec{\nabla}\times\vec{H} = \frac{ic}{\epsilon_0\omega c}\vec{\nabla}\times\vec{H} = \frac{i}{k}\sqrt{\frac{\mu_0}{\epsilon_0}}\vec{\nabla}\times\vec{H}.$$

3. (12) A source of size d is emitting radiation at wavevector k. How do you define the

a) near (static) zone;

$r \ll \lambda$ or $kr \ll 1$.

b) far (radiation) zone?

$r \gg \lambda$ or $kr \gg 1$.

4. (12) The vector potential of an oscillating electric dipole is $\vec{A} = \frac{-i\omega\mu_0 e^{ikr}}{4\pi r}\vec{p}$.

a) What is the definition of $\vec{p}$?

$$\vec{p} = \int d^3x'\, \vec{x}'\, \rho(\vec{x}')$$

b) Express the associated magnetic field $\vec{H}$ at arbitrary radius r. Use $\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}$

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\left[\frac{-i\omega\mu_0 e^{ikr}}{4\pi r}\vec{p}\right] = \frac{-i\omega}{4\pi}\hat{n}\times\vec{p}\frac{\partial}{\partial r}\left[\frac{e^{ikr}}{r}\right] = \frac{-i\omega}{4\pi r}\hat{n}\times\vec{p}\left(ik-\frac{1}{r}\right)e^{ikr}$$

5. (6) If the frequency of an oscillating electric dipole is doubled, by what factor will its total radiated power change?

Power $\propto |\vec{H}|^2 \propto k^4$, so with $k=\frac{\omega}{c}$, the Power will increase by 16x.

6. (8) A radiation source produces certain fields $\vec{E}$ and $\vec{H}$ far from the source. How do you use them to obtain the power radiated per unit solid angle, along a direction $\hat{n}$?

Poynting vector $\vec{S} = \vec{E}\times\vec{H}$, is energy flux, then do time-ave. along $\hat{n}$.

$$\frac{dP}{d\Omega} = \frac{1}{2}r^2\,\mathrm{Re}\{\hat{n}\cdot\vec{E}\times\vec{H}^*\}$$, c.c. needed on $\vec{H}$ for t-averaging.

7. (18) Jackson says the vector potential of an electric quadrupole is

$$\vec{A} = \frac{-\mu_0 c k^2}{8\pi} \frac{e^{ikr}}{r} (1 - \frac{1}{ikr}) \int d^3x' \; \vec{x}' \; (\hat{n} \cdot \vec{x}') \; \rho(\vec{x}').$$

a) Write an expression for its magnetic field in the radiation zone.

At $kr \gg 1$, can replace $\vec{B} = \vec{\nabla} \times \vec{A} \rightarrow ik\hat{n} \times \vec{A}$, and use largest term only.

$\vec{H} = \frac{1}{\mu_0}\vec{B} = \frac{1}{\mu_0} ik\hat{n} \times \vec{A} = \frac{-ick^3}{8\pi} \frac{e^{ikr}}{r} \int d^3x' \; (\hat{n} \times \vec{x}')(\hat{n} \cdot \vec{x}') \rho(\vec{x}')$

b) How is its quadrupole tensor defined?

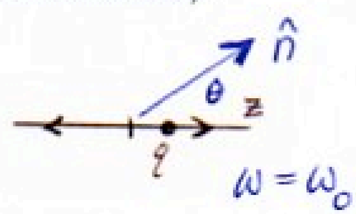
$Q_{\alpha\beta} = \int d^3x' \; (3x'_\alpha x'_\beta - \delta_{\alpha\beta} r^2) \; \rho(\vec{x}')$

c) The total power from this expression is $P = \frac{c^2 Z_0 k^6}{1440\pi} \sum_{\alpha,\beta} |Q_{\alpha,\beta}|^2$. What approximations, if any, have been made to arrive at this formula?

This calculation employed an expansion of $e^{ik|\vec{x}-\vec{x}'|}$, which means, a long wavelength approximation has been applied.

8. (18) For the following sources, describe the predominant type of radiation (multipole and frequency), and the dependence of $dP/d\Omega$ on polar angle θ (between $\hat{n}$ and the z-axis).

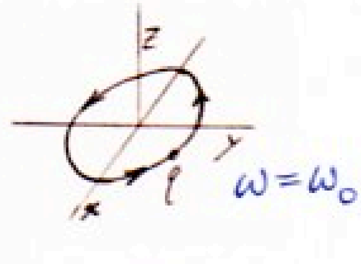
a) A point charge q moving sinusoidally at frequency ω_0 on the z-axis.



It is a linear oscillating electric dipole, $\vec{p} = p\hat{z}$.

$\frac{dP}{d\Omega} \sim |\hat{n} \times \vec{p}|^2 \sim \sin^2\theta$.

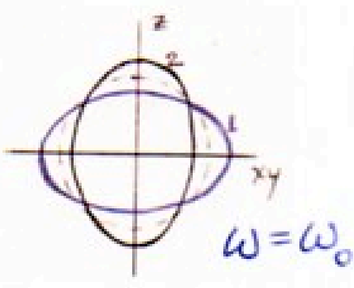
b) A point charge q rotating at angular velocity ω_0 in a circle of radius a in the xy-plane.



It is a rotating electric dipole, or, two dipoles out of phase by $90°$, like $\vec{p} = p_0(\hat{x} + i\hat{y})$

$\frac{dP}{d\Omega} \sim |\hat{n} \times \vec{p}|^2 \sim 1 + \cos^2\theta$

c) A spheroidal charge distribution with azimuthal symmetry around the z-axis, whose shape oscillates between the two extremes 1 and 2 as shown.



Azimuthal symmetry $\Rightarrow$ $Q_{xx} = Q_{yy} = -\frac{1}{2} Q_{zz}$, all oscillate at $\omega = \omega_0$.

$\frac{dP}{d\Omega} \sim |\hat{n} \times \vec{Q}(\hat{n})|^2 \sim \sin^2\theta \cos^2\theta$

9. (12) Give a definition of differential scattering cross section, $d\sigma/d\Omega$. Be as complete as possible.

$\frac{d\sigma}{d\Omega} = \frac{dP_{scatt}/d\Omega}{|\vec{S}_{inc}|}$ where $\vec{S}_{inc} = \frac{1}{2} \mathrm{Re}\{\vec{E}_{inc} \times \vec{H}^*_{inc}\}$ is the incident Poynting vector.

10. (12) What is the definition of a "vector spherical harmonic?" List two orthogonality properties that they have.

$$\vec{X}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L} Y_{\ell m} \qquad \int d\Omega\, \vec{X}^{*}_{\ell m} \cdot \vec{X}_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'}$$

(solution of angular part of Helmholtz eqn.)

$$\int d\Omega\, \vec{X}^{*}_{\ell m} \cdot \vec{\nabla}\times\left[f(r)\vec{X}_{\ell' m'}\right] = 0.$$

11. (8) When an expansion of a circularly polarized plane wave $\vec{E} = E_0(\hat{e}_1 \pm i\hat{e}_2)e^{ikz}$ is made, what possible multipoles (l, m) can be present?

any ℓ, but with $m = \pm 1$ only.

12. (8) The total scattering cross-section of a small dielectric sphere is proportional to what powers of wavevector k and radius a?

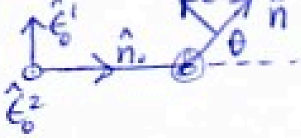
$$\frac{d\sigma}{d\Omega} \sim k^4 a^6 \quad \text{also,} \quad \sigma \sim k^4 a^6$$

13. (12) The differential scattering cross-section of a dielectric sphere is proportional to $|\hat{\epsilon}^{*} \cdot \hat{\epsilon}_0|^2$, where $\hat{\epsilon}_0$ and $\hat{\epsilon}$ are the incident and outgoing polarizations. For unpolarized incident light,

a) At what scattering angle(s) θ (between incident and outgoing wavevectors) does $d\sigma_{\parallel}/d\Omega$ reach any extrema?

$$\hat{\epsilon}\cdot\hat{\epsilon}_0 = \cos\alpha\cos\theta, \quad \frac{d\sigma_{\parallel}}{d\Omega} \sim \cos^2\theta.$$

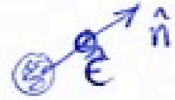
Has extrema at $\theta = 0$, $\theta = \pi/2$.



b) At what scattering angle(s) θ (between incident and outgoing wavevectors) does $d\sigma_{\perp}/d\Omega$ reach any extrema?

$$\hat{\epsilon}\cdot\hat{\epsilon}_0 = \sin\alpha \cdot 1, \quad \frac{d\sigma_{\perp}}{d\Omega} \sim 1.$$

This one has no extrema.



14. (8) How do you define the "relative polarization" of scattered radiation, $\Pi(\theta)$? Be as specific as possible.

$$\Pi(\theta) = \frac{d\sigma_{\perp}/d\Omega - d\sigma_{\parallel}/d\Omega}{d\sigma_{\perp}/d\Omega + d\sigma_{\parallel}/d\Omega}$$

usually would be applied for unpolarized incident radiation.

15. (18) Explain the symbols and application of the following formula:

attenuation coeff. $= \alpha = N\sigma_1 = \frac{2k^4}{3\pi N}|n-1|^2.$ $\quad k = \frac{2\pi}{\lambda}$ = wavevector.

number of scatterers per volume. $\quad$ cross-section for a single scatterer $\quad$ index of refraction of the medium.

It shows how scattering + attenuation increase with k^4 (Rayleigh's Law) and how the effects scale with density N and index of refraction.

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Instructions: Use SI units. Please show the details of your derivations here. Explain your reasoning for full credit. Open-book and 1-page note summary allowed.

1. Linearly polarized plane wave radiation is incident on a free electron of charge $-e$, mass m. The amplitude is small enough so that the motion is nonrelativistic, and determined primarily by the electric field,

$$\vec{E}_{\rm inc} = E_0 \hat{\epsilon}_0 e^{i(k\hat{n}_0\cdot\vec{r}-\omega t)}.$$

   a) (12) Determine the motion of the electron from Newton's 2nd Law (use the harmonic approach, i.e., global $e^{-i\omega t}$ dependences).

   b) (8) Find the time-dependent induced electric dipole moment.

   c) (24) Determine the scattered electric field $\vec{E}_{\rm sc}$ in the radiation zone.

   d) (24) Averaging over incident polarization $\hat{\epsilon}$, find the differential scattering cross sections $d\sigma_{\parallel}/d\Omega$ and $d\sigma_{\perp}/d\Omega$ for scattering within and perpendicular to the scattering plane.

   e) (16) Evaluate the total scattering cross section σ. What value (in meters) of classical electron radius r_e does it imply ($\sigma = \pi r_e^2$)?

a) Newton: $\vec{F} = m\vec{a} = m\ddot{\vec{x}}$ where $\vec{F} = -e\vec{E}_{\rm inc}$.

Assume all quantities vary as $e^{-i\omega t}$, then $\ddot{\vec{x}} = (-i\omega)\vec{x}$. Combining these gives

$$-e\vec{E}_{\rm inc} = -m\omega^2\vec{x}$$

$$\vec{x} = \frac{e}{m\omega^2}\vec{E}_{\rm inc} = \frac{e}{m\omega^2}E_0\hat{\epsilon}_0 e^{i(k\hat{n}_0\cdot\vec{x}-\omega t)}$$

take $k\vec{x}$ as small. due to small amplitude.

$$\boxed{\vec{x}(t) = \frac{eE_0}{m\omega^2}\hat{\epsilon}_0 e^{-i\omega t}}$$

b) Induced dipole, for a single point charge, is just product of $\vec{x}$ and q.

$$\boxed{\vec{p}(t) = -e\vec{x}(t) = \frac{-e^2E_0}{m\omega^2}\hat{\epsilon}_0 e^{-i\omega t}.}$$

c) Radiation zone electric field? At some point $\vec{r}$. $\omega = ck$.

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\,{-i\omega}\vec{p}\,, \quad \vec{H} = \frac{1}{\mu_0}\,ik\hat{n}\times\vec{A} = \frac{ik}{\mu_0}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\,\hat{n}\times\vec{p}$$

$$\vec{H} = \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}\,\hat{n}\times\vec{p}\,,$$ also get $\vec{E}$ from Maxwell-Faraday,

$$\vec{E} = \frac{i}{k}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\vec{\nabla}\times\vec{H} = \frac{i}{k}\sqrt{\frac{\mu_0}{\epsilon_0}}\,ik\hat{n}\times\vec{H} = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\vec{H}\times\hat{n}$$

$$\vec{E}_{sc} = \sqrt{\frac{\mu_0}{\epsilon_0}}\cdot\frac{ck^2}{4\pi}\frac{e^{ikr}}{r}(\hat{n}\times\vec{p})\times\hat{n}$$

or $$\boxed{\vec{E}_{sc} = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{ck^2}{4\pi}\frac{e^{ikr}}{r}\cdot\left(\frac{-e^2E_0}{m\omega^2}\right)(\hat{n}\times\hat{\epsilon}_0)\times\hat{n}\;\cdot\; e^{-i\omega t}}.$$

d) As we're seen $\frac{d\sigma_{\parallel}}{d\Omega} \sim \frac{1}{2}\cos^2\theta$, $\frac{d\sigma_{\perp}}{d\Omega} \sim \frac{1}{2}$. in a previous question, or use basic defn.

$$\frac{d\sigma_{\parallel}}{d\Omega} = \left(\frac{k^2}{4\pi\epsilon_0 E_0}\right)^2\left|\hat{\epsilon}_{\parallel}^{*}\cdot\vec{p}\right|^2 \longrightarrow \left(\frac{k^2}{4\pi\epsilon_0}\right)^2\left(\frac{-e^2}{m\omega^2}\right)^2\cdot\frac{\cos^2\theta}{2}$$

But since $\omega = ck$ this reduces to

$$\boxed{\frac{d\sigma_{\parallel}}{d\Omega} = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{e^2}{mc^2}\right)^2\cos^2\theta.}$$

The $\perp$ x-section is the same except no factor of $\cos^2\theta$.

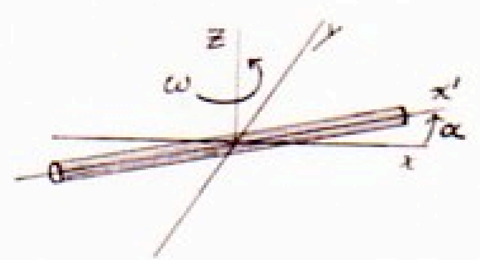
$$\boxed{\frac{d\sigma_{\perp}}{d\Omega} = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{e^2}{mc^2}\right)^2.}$$

e) Total cross-section. $\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2(1+\cos^2\theta)$, $\int d\Omega\,(1+\cos^2\theta) = 4\pi + \frac{4\pi}{3}$.

$$\boxed{\sigma = \frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2\frac{16\pi}{3} = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2} = \pi r_c^2.$$

$$r_c = \sqrt{\frac{\sigma}{\pi}} = 4.6\times10^{-15}\,m\,, \quad \text{where } \sigma = 6.65\times10^{-29}\,m^2$$

2. A cylinder of length d and circular cross-section, radius a, contains a uniform volume charge density ρ. Here we consider the radiation it produces when rotated at angular velocity ω around an axis (z) through its center, perpendicular to the cylinder axis.



a) (20) [Optional!] Show that its electric quadrupole tensor in a $x'y'z'$ coordinate system fixed on the cylinder, with x' along the cylinder axis, and y', z', transverse, is

$$Q' = Q_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \qquad Q_0 = \rho\pi a^2 d \ \left(\frac{d^2}{6} - a^2\right).$$

b) (20) [Also Optional!] The cylinder rotates through angle $\alpha = \omega t$ around the z axis as time progresses. Use the transformation properties for tensors of 2nd rank to show that the time-dependent quadrupole tensor (in lab frame) is

$$Q(t) = Q_0 \begin{pmatrix} \frac{1}{4}(1 + 3\cos 2\omega t) & \frac{3}{4}\sin 2\omega t & 0 \\ \frac{3}{4}\sin 2\omega t & \frac{1}{4}(1 - 3\cos 2\omega t) & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

c) (20) [Start Here] Use this $Q(t)$ to determine the complex quadrupole tensor to use in the analysis of the radiation. What will be the frequency of the emitted radiation?

d) (20) Find the radiated magnetic field $\vec{H}$ in the radiation zone. Give the xyz components of $\vec{H}$ as functions of the angular direction θ, ϕ of unit wave vector $\hat{n}$.

e) (20) Determine the angular distribution of radiated power, as a function of θ, ϕ.

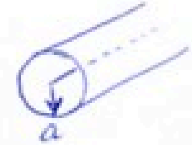
f) (10) At a point on the x-axis in the radiation zone, along what direction is the radiation polarized?

a) $$Q_{x'x'} = \int d^3x' \left(3x'^2 - r'^2\right)\rho = \int_{-d/2}^{d/2} \pi a^2 dx' \left(2x'^2 - y'^2 - z'^2\right)\rho$$

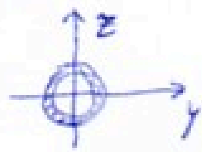
Better to do the following integrals and then combine to get quadrupole tensor.

$$I_{x'} = \pi a^2 \rho \int_{-d/2}^{d/2} dx'\, x'^2 = \pi a^2 \rho \left.\frac{x'^3}{3}\right|_{-d/2}^{d/2} = \pi a^2 d\rho \cdot \frac{d^2}{12}$$

$$I_{y'} = I_{z'} = \rho\int d^3x'\, y'^2 = \rho \int_0^a (2\pi y' dy' \cdot d)\, y'^2 = \pi a^2 d\rho \cdot \frac{a^2}{2}$$



$$\rho\int d^3x'\, r'^2 = \rho\int d^3x' \left(x'^2 + y'^2 + z'^2\right) = I_{x'} + 2I_{y'} = \pi a^2 d\rho \cdot \left(a^2 + \frac{d^2}{12}\right)$$



$$Q_{x'x'} = 2I_{x'} - 2I_{y'} = 2\left(\pi a^2 d\rho\right)\left(\frac{d^2}{12} - \frac{a^2}{2}\right) = \pi a^2 d\rho\left(\frac{d^2}{6} - a^2\right)$$

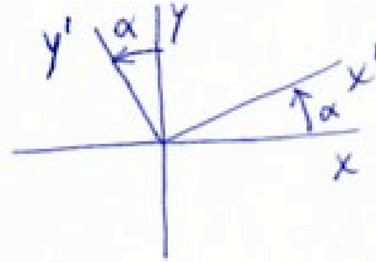
$$Q_{y'y'} = Q_{z'z'} = I_{y'} - I_{x'} = -\frac{1}{2} Q_{x'x'} = -\frac{\pi a^2 d\rho}{2}\left(\frac{d^2}{6} - a^2\right)$$

a) Then in the coords. fixed on the rod, the quadrupole tensor is

$$Q' = \begin{pmatrix} Q_{x'x'} & 0 & 0 \\ 0 & -\frac{1}{2}Q_{x'x'} & 0 \\ 0 & 0 & -\frac{1}{2}Q_{x'x'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} Q_0 , \qquad Q_0 \equiv \pi a^2 d\rho\left(\frac{d^2}{6} - a^2\right).$$

b) Now, of course, the rod is rotating, which makes Q a function of time.
The transformation of the coords. is

$$x = x'\cos\alpha - y'\sin\alpha \qquad xyz = \text{lab}$$
$$y = x'\sin\alpha + y'\cos\alpha \qquad x'y'z' = \text{object.}$$
$$z = z'$$



or $\vec{r} = M\cdot\vec{r}\,'$ is transformation of vectors; $x_\alpha = \frac{\partial x_\alpha}{\partial x'_\beta}x'_\beta = M_{\alpha\beta}\,x_\beta$

Use this M-matrix twice to transform a 2nd rank tensor, applied on each index of $Q_{\alpha\beta}$:

$$Q_{\alpha\beta} = \frac{\partial x_\alpha}{\partial x'_\gamma}\frac{\partial x_\beta}{\partial x'_\delta}Q'_{\gamma\delta} = M_{\alpha\gamma}M_{\beta\delta}Q'_{\gamma\delta} = M_{\alpha\gamma}Q'_{\gamma\delta}M^T_{\delta\beta}$$

transposed matrix.

$$\underset{\text{in lab}}{Q} = M\cdot \underset{\text{in rotating frame fixed on body.}}{Q'}\cdot M^T.$$

$$Q = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} Q_0$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ \frac{1}{2}\sin\alpha & -\frac{1}{2}\cos\alpha & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} Q_0 \qquad \text{with } \alpha = \omega t$$

only this part oscillates

$$= \begin{pmatrix} \frac{1}{4}(1+3\cos 2\alpha) & \frac{3}{4}\sin 2\alpha & 0 \\ \frac{3}{4}\sin 2\alpha & \frac{1}{4}(1-3\cos 2\alpha) & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} Q_0 = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} Q_0 + \begin{pmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{3Q_0}{4}$$

c) Only the oscillatory part of $Q(t)$ produces radiation. Write it in the complex format:

$$Q(t) = \frac{3Q_0}{4}\begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Re}\left\{ \frac{3Q_0}{4}\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{-i2\omega t} \right\}$$

So $\tilde{Q} = \frac{3Q_0}{4}\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The frequency of the emitted radiation is 2ω.

d) To get $\vec{H}$ we first need $\vec{Q} = \tilde{Q}\cdot\hat{n}$, as we need to evaluate

$$\vec{H} = -\frac{1}{3}\frac{ick^3}{8\pi}\frac{e^{ikr}}{r}\,\hat{n}\times(\vec{\tilde{Q}}\cdot\hat{n}) \qquad \text{where} \quad k = \frac{2\omega}{c}.$$

$$\vec{Q} = \tilde{Q}\cdot\hat{n} = \frac{3Q_0}{4}\left[\hat{x}(n_x + in_y) + \hat{y}(in_x - n_y)\right]$$

$$\hat{n}\times\vec{Q} = (n_x\hat{x} + n_y\hat{y} + n_z\hat{z})\times\frac{3Q_0}{4}\left[\hat{x}(n_x+in_y) + \hat{y}(in_x - n_y)\right]$$

$$= \left\{\hat{x}\cdot -n_z(in_x - n_y) + \hat{y}\,n_z(n_x+in_y) + \hat{z}\left[n_x(in_x-n_y) - n_y(n_x+in_y)\right]\right\}\frac{3Q_0}{4}$$

$$= \left\{\hat{x}\cdot -in_z + \hat{y}\,n_z + \hat{z}\cdot i(n_x+in_y)\right\}(n_x+in_y)\frac{3Q_0}{4}.$$

Get the xyz components of $\vec{H}$, with $n_z = \cos\theta$, $n_x + in_y = e^{i\varphi}\sin\theta$

$$H_x = \frac{-ck^3}{24\pi}\frac{e^{ikr}}{r}\cdot\frac{3Q_0}{4}\,n_z(n_x+in_y) = \frac{-ck^3}{32\pi}\frac{e^{ikr}}{r}Q_0\,e^{i\varphi}\sin\theta\cos\theta$$

$$H_y = \frac{-ick^3}{24\pi}\frac{e^{ikr}}{r}\frac{3Q_0}{4}\,n_z(n_x+in_y) = \frac{ck^3}{32\pi}\frac{e^{ikr}}{r}Q_0\,ie^{i\varphi}\sin\theta\cos\theta$$

$$H_z = \frac{ck^3}{24\pi}\frac{e^{ikr}}{r}\frac{3Q_0}{4}(n_x+in_y)^2 = \frac{ck^3}{32\pi}\frac{e^{ikr}}{r}Q_0\,e^{2i\varphi}\sin^2\theta.$$

e) Angular distribution of Radiated Power

$$\frac{dP}{d\Omega} = \frac{1}{2} r^2 \,\mathrm{Re}\{\hat{n}\cdot \vec{E}\times\vec{H}^*\} = \frac{1}{2} r^2 Z_0 |\vec{H}|^2$$

$$\frac{dP}{d\Omega} = \frac{1}{2} r^2 Z_0 \left(\frac{ck^3 Q_0}{32\pi r}\right)^2 \left[\left|-e^{i\varphi}\sin\theta\cos\theta\right|^2 + \left|ie^{i\varphi}\sin\theta\cos\theta\right|^2 + \left|e^{2i\varphi}\sin^2\theta\right|^2 \right]$$

$$= \frac{Z_0}{2}\left(\frac{ck^3 Q_0}{32\pi}\right)^2 \left[2\sin^2\theta\cos^2\theta + \sin^4\theta \right]$$

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0}{2}\left(\frac{ck^3 Q_0}{32\pi}\right)^2 \left(1 - \cos^4\theta\right)}.$$

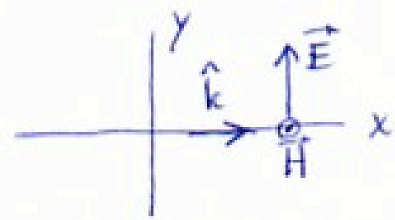
Azimuthally symmetric - no dependence on φ.

f) At a point on the x-axis, we have $\theta = \pi/2$, $\varphi = 0$.

The xyz components of $\vec{H}$ are

$$\vec{H} = \frac{ck^3 e^{ikr}}{32\pi r} Q_0 \left(0, \; 0, \; 1\right)$$

only a z-component.



As $\vec{E}\times\vec{H}$ must be radially along $\hat{k} = \hat{x}$, $\vec{E}$ is in the $\hat{y}$-direction

The radiation is polarized along $\hat{y}$ at this point.