Electrodynamics I — Exam 3 - Part A - Closed Book — KSU 2015/12/18

Name Electrodynamic — Score = 124 / 124 points

Instructions: Use SI units. Where appropriate, define all variables or symbols you use, in words. Try to tell about the physics involved, more than the mathematics, if possible.

1. (10) What expression gives the time-dependent electric field in terms of scalar and vector potentials?

$$\mathbf{E}(\mathbf{x},t) = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A}$$

2. (12) In the presence of any time-dependent sources $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$, what equation is obeyed by the scalar potential when using the Lorenz gauge?

It is a wave equation for scalar potential $\Phi(\mathbf{x},t)$, driven by the charge density $\rho(\mathbf{x},t)$ scaled by ϵ_0; there would be a similar equation for the vector potential $\mathbf{A}(\mathbf{x},t)$:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -\rho(\mathbf{x},t)/\epsilon_0.$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{J}(\mathbf{x},t).$$

3. (10) Consider a 3D wave equation, $\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi = -4\pi f(\mathbf{x},t)$, where $f(\mathbf{x},t)$ is the source that drives some waves $\Psi(\mathbf{x},t)$. Write out the retarded space- and time-dependent Green's function for this equation.

$$G^+(\mathbf{x},t;\mathbf{x}',t') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\delta\left(t'-t+\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

This produces a response at time t from a source at time t', related by

$$t = t' + \frac{|\mathbf{x}-\mathbf{x}'|}{c}$$

that is, the response comes after the propagation time for the waves to arrive at the observer's position.

4. (12) Write out an equation for Poynting's theorem in differential form. Explain in words what each term means physically.

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E} = 0.$$

where u is EM energy density, $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ is the Poynting vector, and $\mathbf{J}$ and $\mathbf{E}$ are current density and electric field. The first term is the increase in EM field energy in a volume element. The second term is the flux of EM energy into that volume element. The third term is the mechanical work done on charges in that volume by the electric field. Poynting's theorem demonstrates that total mechanical plus EM energy is conserved.

5. (12) How does $\mathbf{L}_{\rm em} = \frac{1}{c^2}\int d^3r\ \mathbf{r}\times(\mathbf{E}\times\mathbf{H})$ transform under space inversion? Time inversion?

Since $\mathbf{E}(-\mathbf{r}) = -\mathbf{E}(\mathbf{r})$ (odd), and $\mathbf{H}(-\mathbf{r}) = \mathbf{H}(\mathbf{r})$ (even), while $\mathbf{r}$ itself reverses (odd), $\mathbf{L}_{\rm em}$ is EVEN under spatial inversion.

Since $\mathbf{E}$ is even while $\mathbf{H}$ is odd, their product is ODD under time reversal and so is $\mathbf{L}_{\rm em}$. These are the usual properties for angular momentum.

6. (12) A plane EM wave is traveling in the x-direction in a medium with $\mu = \mu_0$ and $\epsilon = 9\epsilon_0$. With linearly polarized $\mathbf{E}(x,t) = E_0\hat{z}\ \exp[i(kx-\omega t)]$ write an expression for $\mathbf{B}(x,t)$ in this wave.

Faraday's Law gives the relation between $\mathbf{E}$ and $\mathbf{B}$, for harmonic time dependence ($\partial/\partial t \longrightarrow -i\omega$), and with $\nabla \longrightarrow i\mathbf{k}$, and dispersion relation $k = \omega\sqrt{\epsilon\mu}$,

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = i\mathbf{k} \times \mathbf{E} - i\omega \mathbf{B} = 0 \quad \text{or} \quad \mathbf{B} = \frac{1}{\omega}\mathbf{k} \times \mathbf{E} = \sqrt{\epsilon\mu}\,\hat{n} \times \mathbf{E}$$

where $\hat{n} = \mathbf{k}/k = \hat{x}$. The cross product $\hat{n} \times \mathbf{E} \sim \hat{x} \times \hat{z} = -\hat{y}$, and $\sqrt{\epsilon\mu} = 3\sqrt{\epsilon_0\mu_0} = 3/c$. We get

$$\mathbf{B}(x,t) = -\frac{3}{c}\, E_0\, \hat{y}\, \exp[i(kx-\omega t)].$$

7. (12) A plane wave travels in the x-direction: $\mathbf{E}(\mathbf{x},t) = E_0(\hat{y} + i\hat{z})\exp[i(kx-\omega t)]$. Looking into the wave at a fixed point in space, in which direction does the electric field vector rotate (clockwise or couterclockwise)? Which circular polarization is this (right or left)?

Taking a fixed space point ($x = 0$), and looking at the time-dependence, use the implied real part to get the components of $\mathbf{E}$:

$$\mathbf{E} = \mathrm{Re}\left\{E_0(\hat{y} + i\hat{z})\exp(-i\omega t)\right\} = E_0\left[\hat{y}\cos(\omega t) + \hat{z}\sin(\omega t)\right]$$

For small times $t > 0$ the $\hat{z}$ component increases positively, and the rotation of the wave towards the observer is <u>counterclockwise</u>. Pointing your left thumb back towards the source, your left 4 fingers rotate in the counterclockwise sense. This is <u>left circular</u> polarization.

8. (10) When light undergoes total internal reflection at an interface between two optical media with indexes n (incident side) and n' (refraction side), what is required of the incident angle θ?

Use Snell's Law, $n\sin\theta = n'\sin\theta'$ with $\theta' \to 90°$ determines the critical angle. For TIR, the incident angle must be greater than the critical angle, $\theta > \theta_c$, with θ_c defined by $\theta_c = \sin^{-1}(n'/n)$, requiring $n' < n$.

9. (10) Write an expression for the dielectric function $\epsilon(\omega)$ in a plasma.

$$\epsilon(\omega) = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

where the squared plasma frequency ω_p depends on the total volume density of free charges n, by

$$\omega_p^2 = \frac{ne^2}{m\epsilon_0}$$

where m is the carrier mass (usually, the electron mass).

10. (10) What does $\epsilon(\omega)$ imply for EM waves of low frequency traveling in a plasma?

If $\omega < \omega_p$, then the dielectric function becomes negative, $\epsilon < 0$. The wave vector $k = \omega\sqrt{\epsilon\mu}$ becomes pure imaginary, leading to strong damping of the waves entering a plasma, over a short distance. Waves with frequencies below ω_p do not propagate through a plasma.

Part A Score = <u>110</u> /110

Electrodynamics I — Exam 3 - Part B - Open Book — KSU 2015/12/18

Name______________________________________

Instructions: Use SI units. Please show the details of your derivations here. Explain your reasoning for full credit. Open-book only, no notes.

1. (48) Consider the 3D wave equation for an EM field component $\psi(\mathbf{x},t)$,

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -4\pi f(\mathbf{x},t).$$

It is being driven by a source function $f(\mathbf{x}',t') = \delta(x')\delta(y')\delta(t')$, which represents a flash of a line source along the z'-axis at time $t'=0$. You know that the retarded 3D Green's function for this wave equation is

$$G^+(\mathbf{x},t;\mathbf{x}',t') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\delta\left(t'-t+\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right).$$

Consider the EM response in 2D around the line source at a distance $\rho = \sqrt{x^2+y^2}$ from that line source.

(a) (12) Write the general integral expression for the signal $\psi(\mathbf{x},t)$ in terms of an integration over the source function $f(\mathbf{x}',t')$. Do not yet evaluate it.

$$\begin{aligned}
\psi(\mathbf{x},t) &= \int d^3x' \int_{-\infty}^{+\infty} dt'\; G^+(\mathbf{x},t;\mathbf{x}',t') f(\mathbf{x}',t') \\
&= \int d^3x' \int_{-\infty}^{+\infty} dt'\; \frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\delta\left(t'-t+\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\;\delta(x')\delta(y')\delta(t') \\
&= \int d^3x'\; \frac{1}{|\mathbf{x}-\mathbf{x}'|}\;\delta(x')\delta(y') \int_{-\infty}^{+\infty} dt'\;\delta\left(t'-t+\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\delta(t').
\end{aligned}$$

(b) (12) Write out the result after only the integration over source time t' is performed.

Using the last delta function to force only the point $t'=0$ to contribute, in the other delta function,

$$\psi(\mathbf{x},t) = \int d^3x'\; \frac{1}{|\mathbf{x}-\mathbf{x}'|}\;\delta(x')\delta(y')\delta\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right).$$

(c) (24) Now do the integrations over source point $\mathbf{x}'$, including over z'. Determine the final signal $\psi(\mathbf{x},t)$ as a function in the form $\psi(\rho,t)$. Is there some restriction between the observation time t and the radial distance ρ from the source?

Hint: Due to the symmetry, the integration over z' might be easier to do as an integration over source to observer distance $R = |\mathbf{x}-\mathbf{x}'|$. What is the restriction on R?

It simplifies expressions to write in terms of $R=|\mathbf{x}-\mathbf{x}'|$. But the delta functions in x' and y' force $x'=y'=0$, so the only coordinate left to really integrate over is z'. An observation point is at a radial distance ρ from the z-axis. The signal one gets depends only on ρ and the observer time t:

$$\psi(\rho,t) = \int_{-\infty}^{+\infty} dz'\;\frac{1}{R}\,\delta\left(t-\frac{R}{c}\right), \qquad R=\sqrt{\rho^2+(z')^2}.$$

One can see there are two points on the z' axis that satisfy the delta function: $z' = \pm\sqrt{R^2 - \rho^2}$, but *only if* $R > \rho$. If we change to R as the variable of integration, both points from the z' contribute equally, so take twice the contribution of the one at positive z'. First do $\delta(t - R/c) = c\delta(ct - R)$ (check that dimensions are correct). The we use $R\,dR = z'\,dz'$. The integration now becomes

$$\psi(\rho, t) = 2\int_{\rho}^{\infty} dR\,\frac{R}{z'}\,\frac{1}{R}\,c\delta(ct - R) = 2\int_{0}^{\infty} dR\,\Theta(R - \rho)\frac{c}{\sqrt{R^2 - \rho^2}}\,\delta(ct - R)$$

The Heaviside step function enforces $R > \rho$. The delta function picks off $R = ct$ only, giving

$$\psi(\rho, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}.$$

It is clear that the step function does not allow a signal to arrive at the observer until the minimum propagation time $t = \rho/c$ has elapsed. The solution obeys causality!

2. (44) A plane wave of intensity $I_0 = 48.0$ kW/cm^2 and some wave vector of magnitude k is traveling in the $+\hat{z}$ direction, with its electric field vector polarized along $+\hat{x}$. Jackson's Eq. (6.121),

$$\frac{d}{dt}\left(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{field}}\right)_\alpha + \oint_S \sum_\beta (-T_{\alpha\beta})\, n_\beta \, da = 0$$

suggests that $-T_{\alpha\beta}$ is the flux (per unit area per unit time) of α component of field momentum in the β direction. The components of the Maxwell stress tensor are

$$T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2 \mathbf{B}\cdot\mathbf{B}\right)\delta_{\alpha\beta}\right].$$

(a) (16) For the given plane wave, determine how $(-T_{zz})$ is related to the z-component of the electromagnetic momentum density, $\mathbf{g} = (\mathbf{E}\times\mathbf{H})/c^2$.

Assuming $\cos(kz-\omega t)$ time dependence, $\mathbf{k} = \omega\sqrt{\epsilon_0\mu_0}\,\hat{\mathbf{z}}$, and $\hat{n} = \mathbf{k}/\mathbf{k} = \hat{\mathbf{z}}$, we can take the electric and magnetic fields of the plane wave as

$$\mathbf{E} = E_0\hat{x}\cos(kz-\omega t), \qquad \mathbf{B} = \sqrt{\epsilon_0\mu_0}\,\hat{n}\times\mathbf{E} = \frac{E_0}{c}\hat{y}\cos(kz-\omega t).$$

The amplitude of magnetic induction is $B_0 = E_0/c$. Other components of $\mathbf{E}$ and $\mathbf{B}$ are zero. Now using $\alpha = \beta = z$, we get the relevant component of the stress tensor,

$$T_{zz} = \epsilon_0\left[E_z^2 + c^2 B_z^2 - \frac{1}{2}\left(E_0^2 + c^2 B_0^2\right)\cos^2(kz-\omega t)\right] = -\epsilon_0\, E_0^2\cos^2(kz-\omega t).$$

Compare the momentum density, using $c^{-2} = \epsilon_0\mu_0$,

$$g_z = \frac{1}{c^2}\left(\mathbf{E}\times\mathbf{H}\right)_z = \frac{1}{c^2}\,E_x\frac{B_y}{\mu_0} = \frac{1}{c^2}\,E_0\frac{E_0}{c\mu_0}\cos^2(kz-\omega t) = \frac{\epsilon_0}{c}E_0^2\cos^2(kz-\omega t).$$

Comparing the results show that

$$-T_{zz} = c g_z.$$

In other words, $-T_{zz}$ really is the flux of momentum per area per time in the waves. That is the same as the pressure (force per area) in the waves if they are absorbed by some surface. Multiplying the momentum volume density g_z by the speed of light converts it into the flux of momentum, or pressure in the z direction.

Note: the solution can also be found by doing the time averaging of harmonic fields, including a complex conjugate on one quantity of a quadratic pair, and a factor of 1/2.

(b) (16) If the given wave is incident on a perfectly conducting mirror at normal incidence, determine the radiation pressure on the mirror, in terms of I_0 and in N m^{-2}.

If the waves are reflected from a perfect conductor, the wave momentum is reversed, which creates a pressure equal to *twice* the momentum flux. The intensity is the time averaged Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$, which is half of its peak value,

$$I = \langle\mathbf{S}\rangle\cdot\hat{n} = \frac{1}{2}E_0 H_0 = \frac{1}{2}E_0\frac{E_0}{c\mu_0} = \frac{E_0^2}{2c\mu_0}.$$

Then the pressure produced by total reflection is

$$P = 2c\langle g_z\rangle = 2c\frac{\langle\mathbf{E}\times\mathbf{H}\rangle}{c^2} = \frac{2I}{c}.$$

The number is

$$P = \frac{2\times 48\times 1000\times(100)^2\text{W m}^{-2}}{3.00\times 10^8 \text{ m s}^{-1}} = 3.20 \text{ N m}^{-2}.$$

(c) (12) How is the answer to part (b) changed if the angle of incidence is $\theta = 60°$?

If the waves are incident at angle θ, then the component of momentum perpendicular to the surface is proportional to $\cos\theta$. In addition, the waves are incident on an area that is increased by the factor $1/\cos\theta$. The pressure is thus changed by the factor $\cos^2\theta$,

$$P = \frac{2c\langle g_z\rangle \cos\theta}{1/\cos\theta} = \frac{2I}{c}\cos^2\theta.$$

For the case $\cos\theta = \cos 60° = \frac{1}{2}$, the pressure is reduced to $\frac{1}{4}$ of that at normal incidence, then $P = 0.80\ \mathrm{N\ m^{-2}}$.

You could also arrive at this result by finding $-T_{nn}$ where n refers to the component perpendicular to the reflecting surface.

3. (48) The imaginary part of a dielectric function is known to be

$$\frac{\epsilon_I(\omega)}{\epsilon_0} = \frac{\gamma\omega}{\omega^2+\gamma^2}.$$

(a) (16) Apply the Kramers-Kronig relations to obtain the real part of $\epsilon(\omega)$.

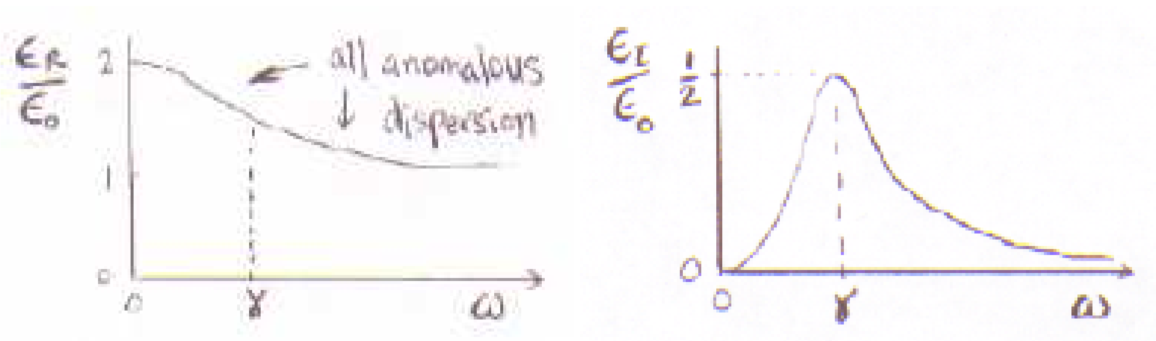
Use the simplest form, with the principal valued integral, closing the contour in the upper half plane:

$$\frac{\epsilon_R(\omega)}{\epsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz\, \frac{\epsilon_I(z)}{z-\omega} = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz\, \frac{1}{z-\omega}\, \frac{\gamma z}{z^2+\gamma^2}$$

$$= \frac{1}{\pi}\left[\pi i \text{ res}(\omega) + 2\pi i \text{ res}(i\gamma)\right] = i\left[\frac{\gamma\omega}{\omega^2+\gamma^2} + \frac{2\gamma\, i\gamma}{(i\gamma-\omega)(2i\gamma)}\right] = \frac{\gamma^2}{\omega^2+\gamma^2}$$

$$\frac{\epsilon_R(\omega)}{\epsilon_0} = 1 + \frac{\gamma^2}{\omega^2+\gamma^2}$$

(b) (16) Sketch the real and imaginary parts of $\epsilon(\omega)$ vs. ω, and identify the regions of normal and anomalous dispersion.



(c) (16) Determine the complex function $\epsilon(\omega)$ and locate its poles in the complex ω-plane. Do its poles occur in the region associated with causal response?

Adding the real and imaginary parts give easily

$$\frac{\epsilon(\omega)}{\epsilon_0} = \frac{\epsilon_R(\omega) + i\,\epsilon_I(\omega)}{\epsilon_0} = 1 + \frac{\gamma^2}{\omega^2+\gamma^2} + i\,\frac{\gamma\omega}{\omega^2+\gamma^2}$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{i\gamma(\omega - i\gamma)}{\omega^2+\gamma^2} = 1 + \frac{i\gamma}{\omega + i\gamma}.$$

The Only pole is at $\omega = -i\gamma$, in the lower half plane. Thus, $\epsilon(\omega)$ is analytic in the upper half plane, as required by causality.

Part B Score = 140 /140