Electrodynamics I — Exam 2 - Part A - Closed Book — KSU 2015/11/06

Name Electrodynamic — Score = 124 / 124 points

Instructions: Use SI units. Where appropriate, define all variables or symbols you use, in words. Try to tell about the physics involved, more than the mathematics, if possible.

1. (8) A charge density $\rho(\mathbf{x})$ is invariant when the system is rotated through any angle around the z-axis. How can you write the general solution of Poisson's equation for the potential $\Phi(\mathbf{x})$ in spherical coordinates in this situation?

The problem has azimuthal symmetry; there is no dependence on ϕ. Then the only spherical harmonics are those with $m = 0$, which are equivalent to the Legendre Polynomials. Include also the correct radial dependence.

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

The expansion coefficients can be determined, for example, by finding the potential on the z-axis and expanding that result in z.

2. (8) Write out the orthogonality condition for the Legendre polynomials $P_l(\cos\theta)$, when defined so that $P_l(0) = 1$ for all l.

Given without any proof,

$$\int_{-1}^{+1} d(\cos\theta) P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}.$$

3. (8) An electric dipole $\mathbf{p}$ at position $\mathbf{x}_0$ is exposed to an external electric field $\mathbf{E}(\mathbf{x})$. Write an expression for the interaction energy of the dipole with this field.

In an external field from some outside source, the dipole tends to align with the field. The interacton energy is

$$W = -\mathbf{p} \cdot \mathbf{E}.$$

This is minimized when $\mathbf{p}$ is parallel to $\mathbf{E}$.

4. (12) The electric potential of an electric dipole $\mathbf{p}$ at the origin is $\Phi(\mathbf{x}) = k\frac{\mathbf{p}\cdot\mathbf{x}}{r^3}$. From this show how to write an expression for the electric field $\mathbf{E}(\mathbf{x})$ at point $\mathbf{x}$ due to an electric dipole at some position $\mathbf{x}_0$.

For a dipole at the origin, the electric field is found from a negative gradient:

$$\mathbf{E}(\mathbf{x}) = -\vec{\nabla}\Phi = -k\left[(\mathbf{p}\cdot\mathbf{x})\vec{\nabla}\left(\frac{1}{r^3}\right) + \frac{\mathbf{p}\cdot\vec{\nabla}\mathbf{x}}{r^3}\right] = k\left[\frac{3(\mathbf{p}\cdot\mathbf{x})\mathbf{x}}{r^5} - \frac{\mathbf{p}}{r^3}\right]$$

The origin can be shifted by letting $\mathbf{x} \to \mathbf{x} - \mathbf{x}_0$. This gives

$$\mathbf{E} = k\left[\frac{3(\mathbf{p}\cdot\hat{n})\hat{n} - \mathbf{p}}{|\mathbf{x} - \mathbf{x}_0|^3}\right].$$

The unit vector $\hat{n}$ points from $\mathbf{x}_0$ towards $\mathbf{x}$.

5. (8) Give a definition of electric polarization $\mathbf{P}$ of a medium.

Electric polarization is the electric dipole moment per unit volume. To make the macroscopic definition one needs to average over some small volume ΔV, and count the dipoles within that volume. Then

$$\mathbf{P}(\mathbf{x}) = \frac{1}{\Delta V} \sum_i \mathbf{p}_i$$

where the $\mathbf{p}_i$ are only those in the considered small volume around a point $\mathbf{x}$.

6. (8) How is electric displacement $\mathbf{D}$ of a medium defined?

The electric displacement is a combination of electric field and the electric polarization,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}.$$

This comes about because polarization generates a bound volume charge density $\rho_b = -\vec{\nabla} \cdot \mathbf{P}$, that contributes also to Gauss' Law,

$$\vec{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho + \rho_b) = \frac{1}{\epsilon_0}(\rho - \vec{\nabla} \cdot \mathbf{P}) \quad \Longrightarrow \quad \vec{\nabla} \cdot \mathbf{D} = \rho.$$

Then Gauss' Law can be written to depend only the free charges when the displacement is used.

7. (8) What is a formula that gives the bound charge density ρ_b inside a dielectric medium?

This was answered in the previous question, $\rho_b = -\vec{\nabla} \cdot \mathbf{P}$. That comes about because any dipoles produce a distant electric dipole potential,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\mathbf{P}(\mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$

Some vector calculus identities give

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \vec{\nabla}' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \cdot \mathbf{P}(\mathbf{x}') = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{-\vec{\nabla}' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

from which ρ_b can be read off in the numerator of the integrand.

8. (8) A dielectric medium has an electric susceptibility tensor with Cartesian components χ_{ij}, where i, j correspond to x, y, z. How is χ_{ij} used or what does it mean?

The susceptibility tensor gives the relation between the electric polarization and an applied electric field,

$$P_i = \epsilon_0 \chi_{ij} E_j.$$

As a result of that, the displacement and permeability are determined by χ,

$$D_i = \epsilon_{ij} E_j = \epsilon_0 E_i + P_i = \epsilon_0 (\delta_{ij} + \chi_{ij}) E_j \quad \Longrightarrow \quad \epsilon_{ij} = \epsilon_0 (\delta_{ij} + \chi_{ij}).$$

9. (8) A current density all over space is given by $\mathbf{J}(\mathbf{x})$. Write an integral expression that gives the magnetic induction $\mathbf{B}(\mathbf{x})$ due to that current density.

The magnetic induction is given by the Biot-savart Law

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}.$$

10. (8) Write an integral expression for the vector potential $\mathbf{A}(\mathbf{x})$ due to a current density $\mathbf{J}(\mathbf{x})$.

To make $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$ consistent with the Biot-Savart law one can show that

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.$$

11. (16) A localized current density $\mathbf{J}(\mathbf{x})$ produces a magnetic dipole moment $\mathbf{m}$.

(a) (8) Write an integral expression for $\mathbf{m}$ in terms of $\mathbf{J}$.

$\mathbf{m}$ will be the volume integral of the magnetic moment density,

$$\mathbf{m} = \frac{1}{2} \int d^3x' \, \mathbf{x}' \times \mathbf{J}(\mathbf{x}')$$

where $\mathbf{x}'$ is measured from some center within the current density.

(b) (8) Write an expression for the vector potential produced by $\mathbf{m}$.

This is analogous to the scalar potential of an electric dipole, but in a vector form,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{x}}{r^3}.$$

12. (8) Give a definition of the magnetization $\mathbf{M}(\mathbf{x})$ of a magnetic material.

The magnetization is the magnetic dipole moment per unit volume, in analogoy with the polarization being the electric dipole moment per unit volume,

$$\mathbf{M}(\mathbf{x}) = \frac{1}{\Delta V} \sum_i \mathbf{m}_i$$

where the sum is over the dipoles within some small volume centered on point $\mathbf{x}$.

13. (8) How is an effective bound current density derived from some magnetization $\mathbf{M}(\mathbf{x})$?

This is also the vector generalization of the bound charge density, i.e.,

$$\mathbf{J}_b(\mathbf{x}) = \vec{\nabla} \times \mathbf{M}(\mathbf{x}).$$

14. (8) How is the magnetic field $\mathbf{H}$ defined?

The magnetic field is a derived field from the magnetic induction and the magnetization,

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

This results because either free or bound currents can generate $\mathbf{B}$, according to a generalized Ampere's Law,

$$\vec{\nabla} \times \mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J}_b) = \mu_0(\mathbf{J} + \vec{\nabla} \times \mathbf{M})$$

whose re-arrangement gives the result

$$\vec{\nabla} \times \mathbf{H} = \mathbf{J}$$

with $\mathbf{H}$ as stated above.

Part A Score = __________/124

Electrodynamics I | Exam 2 - Part B - Open Book | KSU 2015/11/06

Name Electrodynamic | Score = 132 / 132 points

Instructions: Use SI units. Please show the details of your derivations here. Explain your reasoning for full credit. Open-book only, no notes.

1. (36) Inside a sphere of radius a the electric permittivity is a uniform value ϵ_1. Outside there is a another medium with uniform electric permittivity ϵ_2. A point charge q is placed at the center of the sphere (the origin).

(a) (12) How large is the electric field at $r = 2a$?

The point $r = 2a$ is outside the sphere, where the permittivity is ϵ_2 and there are no free charges. The electric field in the region comes from a potential Φ_2 satisfying a Laplace equation, $\nabla^2\Phi_2 = 0$. Due to the spherical symmetry, it must be of monopole form with a radial electric field,

$$\Phi_2 = \frac{B}{r} \quad \Longrightarrow \quad \mathbf{E}_2 = -\vec{\nabla}\Phi_2 = \frac{B\hat{r}}{r^2}.$$

where B is a constant. Inside the sphere, the potential Φ_1 solves a Poisson equation,

$$\nabla^2\Phi_1 = -\rho/\epsilon_1 = -q\delta(\mathbf{r})/\epsilon_1.$$

The interior solution and its radial field are

$$\Phi_1 = \frac{q}{4\pi\epsilon_1 r} \quad \Longrightarrow \quad \mathbf{E}_1 = -\vec{\nabla}\Phi_1 = \frac{q\hat{r}}{4\pi\epsilon_1 r^2}.$$

The radial component of $\mathbf{D} = \epsilon\mathbf{E}$ must be continuous at $r = a$. Matching the two solutions gives

$$\epsilon_1 E_1(a) = \epsilon_2 E_2(a) \quad \Longrightarrow \quad \frac{q}{4\pi} = \epsilon_2 B \quad \Longrightarrow \quad B = \frac{q}{4\pi\epsilon_2} \quad \Longrightarrow \quad \mathbf{E}_2 = \frac{q\hat{r}}{4\pi\epsilon_2 r^2}.$$

This was all fairly obvious. Then finally the field strength at $r = 2a$ is

$$E_2(2a) = \frac{q}{4\pi\epsilon_2(2a)^2} = \frac{q}{16\pi\epsilon_2 a^2}.$$

(b) (12) How large is the free surface charge density at $r = a$?

At the boundary from the interior medium ϵ_1 to the exterior medium ϵ_2 there was no mention of free charges present. A discontinuity in E does not imply any free charges. The radial component of $\mathbf{D}$ is continuous, and due to $\vec{\nabla}\cdot\mathbf{D} = \rho$, application of Gauss' Law at the surface $r = a$ shows that there is no free charge density there. A Gaussian spherical surface of radius $r = a - \delta a$ will enclose the same amount of free charge as one of radius $r = a + \delta a$, where $\delta a \ll a$ is a small parameter.

(c) (12) How large is the bound surface charge density at $r = a$?

The bound surface charge density σ_b can be found by applying Gauss' Law to the definition for bound charge density, $\vec{\nabla}\cdot\mathbf{P} = -\rho_b$. Using a spherical shell of inner radius $r_1 = a - \delta a$ and outer radius $r_2 = a + \delta a$, one has

$$\oint \mathbf{P}\cdot\hat{n}\, dA = -q_{\text{enc}} \quad \Longrightarrow \quad 4\pi a^2(\mathbf{P}_2 - \mathbf{P}_1)\cdot\hat{r} = -4\pi a^2\sigma_b \quad \Longrightarrow \quad \sigma_b = -(\mathbf{P}_2 - \mathbf{P}_1)\cdot\hat{r}.$$

From $\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$, the polarization on each side is $\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E}$. Then the bound surface charge density is

$$\sigma_b = -(\epsilon_2 - \epsilon_0)E_2(a) + (\epsilon_1 - \epsilon_0)E_1(a) = \epsilon_0[E_2(a) - E_1(a)]$$

$$\sigma_b = \epsilon_0\left(\frac{q}{4\pi\epsilon_2 a^2} - \frac{q}{4\pi\epsilon_1 a^2}\right) = \frac{\epsilon_0 q}{4\pi a^2}\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right).$$

If $\epsilon_1 > \epsilon_2$ and $q > 0$, then $\sigma_b > 0$.

2. (48) Consider the problem of finding an electrostatic potential $\Phi(\rho, \phi, z)$ in the region $z > 0$, above an infinite plane, on which the potential given in cylindrical coordinates is $V(\rho, \phi)$.

(a) (16) Using cylindrical coordinates, write an integral expression for the potential $\Phi(\rho, \phi, z)$ above the plane in terms of Bessel functions $J_m(x)$. It may contain some unknown expansion coefficients that depend on m.

The solution region is $z > 0$. Separation of variables applied to the Poisson equation $\nabla^2\Phi = -\rho/\epsilon_0$ in cylindrical coordinates shows that a basic solution, before applying boundary conditions, is

$$\Phi(\rho, \phi, z) \sim e^{-kz} e^{im\phi} J_m(k\rho).$$

We are assuming no charges in the region. Normally the values of k and m are determined by boundary conditions. But with no boundary in the ρ-direction, k takes a continuum of values, resulting in an integral over Bessel functions and the parameter k. For solutions to be single-valued as functions of ϕ, m must be integers. For finite potential as $z \to \infty$ one needs $k \geq 0$. Then including some expansion coefficients $A_m(k)$, we have a form of the solution

$$\Phi(\rho, \phi, z) = \int_0^\infty dk \sum_{m=0}^{\infty} A_m(k) e^{-kz} e^{im\phi} J_m(k\rho).$$

The $A_m(k)$ must be determined from the boundary conditions.

(b) (16) Using appropriate orthogonality conditions, determine an integral expression for those expansion coefficients.

The boundary condition is given on the plane $z = 0$ as $\Phi(\rho, \phi, z = 0) = V(\rho, \phi)$. This is the equation,

$$V(\rho, \phi) = \int_0^\infty dk\ A_m(k) e^{im\phi} J_m(k\rho).$$

One can multiply on both sides by another (complex conjugate) of the basis functions and then integrate over ρ, ϕ.

$$\int_0^{2\pi} d\phi \int_0^\infty \rho\, d\rho\, e^{-im'\phi} J_{m'}(k'\rho) V(\rho, \phi) = \int_0^{2\pi} d\phi \int_0^\infty \rho\, d\rho\, e^{-im'\phi} J_{m'}(k'\rho) \int_0^\infty dk \sum_{m=0}^{\infty} A_m(k) e^{im\phi} J_m(k\rho)$$

On the RHS the integration over ϕ is 2π times a Kronecker delta function in (m, m'),

$$\int_0^{2\pi} d\phi\ e^{-im'\phi} e^{im\phi} = 2\pi \delta_{m,m'}$$

which forces $m' = m$. The remaining integration over ρ is the orthogonality integral for the Bessel functions (see Problem 3.16),

$$\int_0^\infty \rho\, d\rho\ J_m(k\rho) J_m(k'\rho) = \frac{1}{k} \delta(k - k').$$

Using these above now gives

$$\int_0^{2\pi} d\phi \int_0^\infty \rho\, d\rho\ e^{-im\phi} J_m(k'\rho) V(\rho, \phi) = \int_0^\infty dk\ \frac{2\pi}{k} \delta(k - k') A_m(k) = \frac{2\pi}{k'} A_m(k').$$

Renaming $k' \to k$, the expansion coefficients are

$$A_m(k) = \frac{k}{2\pi} \int_0^{2\pi} d\phi \int_0^\infty \rho\, d\rho\ e^{-im\phi} J_m(k\rho) V(\rho, \phi).$$

(c) (16) Now consider that the potential given on the plane is circularly symmetric,

$$V(\rho, \phi) = \begin{cases} V_0, & \rho < a \\ 0, & \rho > a \end{cases}$$

If possible, evaluate the expansion coefficients.

Hint: The recursion relations for the Bessel functions will be helpful.

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x).$$

$$J_{n-1}(x) - J_{n+1}(x) = 2\frac{d}{dx} J_n(x).$$

This given potential has no dependence on ϕ (azimuthal symmetry). One can see evaluating the $A_m(k)$ integral that only $m = 0$ will have nonzero values. So now we just need to find

$$A_0(k) = \frac{k}{2\pi} \int_0^{2\pi} d\phi \int_0^a \rho\, d\rho\ J_0(k\rho)\, V_0 = kV_0 \int_0^a \rho\, d\rho\ J_0(k\rho) = \frac{V_0}{k} \int_0^{ka} dx\ x\, J_0(x).$$

In the last step we take $x = k\rho$. Now the stated recursion relations can be used. With index $n = 0$ and $n = 1$, they give

$$J_{-1} = -J_1, \qquad J_{-1} - J_1 = 2\frac{d}{dx} J_0. \quad \Longrightarrow \quad J_1 = -J_0'.$$

$$J_0 + J_2 = \frac{2}{x} J_1, \qquad J_0 - J_2(x) = 2\frac{d}{dx} J_1. \quad \Longrightarrow \quad J_0 = \frac{1}{x} J_1 + J_1'.$$

Then using these can show that the integrand we have is a perfect differential,

$$xJ_0 = J_1 + xJ_1' = -J_0' - xJ_0'' = -\frac{d}{dx}\left(xJ_0'\right).$$

Then we get quickly

$$A_0(k) = \frac{V_0}{k} \int_0^{ka} dx\ \frac{-d}{dx}\left(xJ_0'\right) = -\frac{V_0}{k}\left(ka\, J_0'(ka)\right) = aV_0\, J_1(ka).$$

3. (48) A magnetic field is created by a localized distribution of permanent magnetization $\mathbf{M}$, without free currents, $\mathbf{J}$=0. Thus, assume $\mathbf{M}$ can be derived from a magnetic scalar potential by $\mathbf{H} = -\vec{\nabla}\Phi_M$. The total magnetic energy to assemble such a system is to be found.

(a) (16) Use the appropriate Maxwell's equations and get the differential equation that Φ_M obeys.

There are no magnetic monopoles, from $\vec{\nabla}\cdot\mathbf{B} = 0$, and we also have Ampere's Law, $\vec{\nabla}\times\mathbf{H} = \mathbf{J}$. But there are no free currents. We do have the consituitive relation, $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(-\vec{\nabla}\Phi_M + \mathbf{M})$. Then a little algebra gives,

$$\vec{\nabla}\cdot\mathbf{B} = \mu_0\left(-\vec{\nabla}\cdot\vec{\nabla}\Phi_M + \vec{\nabla}\cdot\mathbf{M}\right) = 0 \quad \Longrightarrow \quad \nabla^2\Phi_M = \vec{\nabla}\cdot\mathbf{M}.$$

That displays the effective magnetic charge density, $\rho_M = -\vec{\nabla}\cdot\mathbf{M}$.

(b) (16) Show that the field energy integral over all space is zero:

$$W_0 = \frac{1}{2}\int d^3x\, \mathbf{B}\cdot\mathbf{H} = 0.$$

Integration by parts or vector integral calculus identities might be helpful.

Use the scalar potential to express the fields: $\mathbf{H} = -\vec{\nabla}\Phi_M$ and $\mathbf{B} = \mu_0(-\vec{\nabla}\Phi_M + \mathbf{M})$. Integrate by parts:

$$W_0 = \frac{1}{2}\int d^3x\, \mathbf{B}\cdot(-\vec{\nabla}\Phi_M) = -\frac{\mu_0}{2}\int d^3x\, \left[\vec{\nabla}\cdot(\mathbf{B}\Phi_M) - (\vec{\nabla}\cdot\mathbf{B})\Phi_M\right]$$

The last part is zero because $\vec{\nabla}\cdot\mathbf{B} = 0$. The first part can be transformed by the divergence theorem to a surface integral:

$$W_0 = -\frac{\mu_0}{2}\oint_S da\, \hat{\mathbf{n}}\cdot\mathbf{B}\Phi_M = 0.$$

Only dipole fields can be present at large r. The surface integral is zero because the surface is taken at large r, where $\mathbf{M} = 0$, and where $\Phi_M \to 0$ faster than $1/r$ and $\mathbf{B}$ is at least as fast as $1/r^2$.

(c) (16) Combine the integral in (b) with the interaction of the dipoles in a field, W_{int}, as would be based on a single-dipole interaction energy,

$$u = -\mathbf{m}\cdot\mathbf{B}.$$

Hint: Turn this into an integral W_{int} over the magnetization $\mathbf{M}$. Consider the process of starting with $\mathbf{M}$=0 everywhere and then bringing it to its final values, understanding that $\mathbf{H}$ and $\mathbf{B}$ are being generated by $\mathbf{M}$.

Show that the total system energy is

$$W = W_0 + W_{\text{int}} = \frac{1}{2}\mu_0\int d^3x\, \left(\mathbf{H}^2 - \mathbf{M}^2\right)$$

One needs to imagine turning on $\mathbf{M}$, starting from zero. The increment of interaction energy due to volume element d^3x is

$$\delta(\text{energy}) = \delta U_{\text{int}}\, d^3x = -(\delta\mathbf{M})\cdot\mathbf{B}\, d^3x \quad \Longrightarrow \quad dW_{\text{int}} = -\int_0^{\mathbf{M}}(\delta\mathbf{M})\cdot\mathbf{B}\, d^3x = -\frac{1}{2}\mathbf{M}\cdot\mathbf{B}\, d^3x$$

where U refers to energy per volume, and we integrated the $(\delta \mathbf{M})$ to its final value. This is integrated over space (where $\mathbf{M} \neq 0$) to get the total interaction,

$$W_{\rm int} = -\frac{1}{2} \int d^3x \, \mathbf{M} \cdot \mathbf{B}.$$

Combining with the "zero" integral from (b), we have

$$W = W_0 + W_{\rm int} = \frac{1}{2} \int d^3x \, (\mathbf{B} \cdot \mathbf{H} - \mathbf{B} \cdot \mathbf{M}) = \frac{1}{2} \int d^3x \, \mathbf{B} \cdot (\mathbf{H} - \mathbf{M}).$$

Now with relation, $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$,

$$W = \frac{1}{2} \int d^3x \, \mu_0(\mathbf{H} + \mathbf{M})(\mathbf{H} - \mathbf{M}) = \frac{\mu_0}{2} \int d^3x \, (\mathbf{H}^2 - \mathbf{M}^2).$$

Which is the desired result.

FYI, this can also be written as a combination

$$W = -\frac{\mu_0}{2} \int d^3x \, \mathbf{M} \cdot \mathbf{H} - \frac{\mu_0}{2} \int d^3x \, \mathbf{M}^2.$$

The first integral can depend on the orientation of a magnetization relative to the $\mathbf{H}$ field. The second integral is more of an intrinsic quantity that does not depend directly on $\mathbf{H}$. It could be a constant for a permanent magnet, once magnetized.

Part B Score = __________/132