Electrodynamics I — Midterm Exam - Part A - Closed Book — KSU 2015/09/25

Name Electrodynamic — Score = 120 / 120 points

Instructions: Use SI units. Where appropriate, define all variables or symbols you use, in words. Try to tell about the physics involved, more than the mathematics, if possible.

1. (10) Write Gauss' Law in integral form. State its physical significance.

$$\oint_S da\,\mathbf{E}\cdot\mathbf{n} = \frac{1}{\epsilon_0}\int_V d^3x\,\rho(\mathbf{x}) \quad \text{or} \quad \Phi_E = \frac{1}{\epsilon_0}Q_{\text{enc}}.$$

$\mathbf{E}$=electric field, $\mathbf{n}$=outward normal unit vector, ρ=volume charge density, ϵ_0=permittivity of vacuum, Φ_E=electric flux out of the volume, Q_{enc}=net charge enclosed in volume. It exemplifies how electric flux (LHS integral) is produced by electric charge (RHS integral) within a volume.

2. (10) A fixed electric charge density $\rho(\mathbf{x})$ exists in space with no boundaries. Write an expression for the **electric field** that it produces at a field point $\mathbf{x}$.

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3},$$

where $\mathbf{x}'$ is a source point and $\mathbf{x}$ is the field point where $\mathbf{E}$ is being calculated. The integral is over all of space.

3. (10) A fixed electric charge density $\rho(\mathbf{x})$ exists in space with no boundaries. Write an expression for the **electrostatic potential** that it produces at a field point $\mathbf{x}$.

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|},$$

where $\mathbf{x}'$ is a source point and $\mathbf{x}$ is the field point where the potential Φ is being calculated. The integral is over all of space.

4. (10) Write an expression that gives the electrostatic field energy density in vacuum.

$$u_E = \frac{1}{2}\epsilon_0|\mathbf{E}|^2,$$

where u_E is the energy density in joules/meter^3. This should be integrated over space to get the total electrostatic energy in the field.

5. (10) From a solution for electric potential $\Phi(\mathbf{x})$, how can you obtain the surface charge density σ on a conductor boundary? Let $\hat{n}$ be the unit vector pointing perpendicularly outward from the conductor.

You can use Gauss's Law in integral form on small pillbox at the surface, and get

$$E_n\Delta A = \frac{1}{\epsilon_0}\sigma\Delta A$$

where $E_n = \mathbf{E}\cdot\hat{n}$ is the component of $\mathbf{E}$ outward from the conducting surface. This can be solved and also related to the electric potential,

$$\sigma = \epsilon_0 E_n = -\epsilon_0\left.\frac{\partial\Phi}{\partial n}\right|_S.$$

The derivative is along the axis outward from the surface, at the surface itself.

6. (10) From the previous two answers, how do you write the force per unit area caused by the electric field on a conductor surface? Express the result as a vector.

The electrostatic energy in a small volume of thickness Δx and area A just outside a conducting surface can be used to get the force:

$$\Delta U_E = u_E \Delta V = u_E A \Delta x \quad \Longrightarrow \quad F = \frac{\Delta U_E}{\Delta x} = u_E A.$$

Then the magnitude of force per unit area is the same as $F/A = u_E = \frac{1}{2}\epsilon_0 |\mathbf{E}|^2$. The direction is parallel to $\Delta q\, \mathbf{E}$, and from Question 5 with $\sigma = \epsilon_0 E_n$ we can write

$$\mathbf{F}/A = \frac{1}{2}\sigma \mathbf{E}.$$

Curiously this is always outward from the surface, regardless of the sign of σ.

7. (10) Write Poisson's equation for electrostatics. Explain what it is good for.

$$\nabla^2 \Phi(\mathbf{x}) = -\rho(\mathbf{x})/\epsilon_0.$$

which is derived from the differential form of Gauss' Law and the electrostatic assumption that $\mathbf{E}(\mathbf{x}) = -\vec{\nabla}\Phi$. For a given volume charge density $\rho(\mathbf{x})$, the solution will give the electrostatic potential, which then ultimately describes the electric field distribution.

8. (10) Write a differential equation that a Green function $G(\mathbf{x}, \mathbf{x}')$ for Poisson's equation must satisfy, for *Dirichlet* boundary conditions. Include a statement of the boundary conditions.

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}'),$$

where $\mathbf{x}'$ is the source point, $\mathbf{x}$ is the observation point or field point where a response (the potential field) is measured. The boundary condition is that the Green's function vanishes on the boundary:

$$G(\mathbf{x}, \mathbf{x}') = 0 \quad \forall\ \mathbf{x} \in S,\ \mathbf{x}' \in S.$$

9. (10) A problem has *Dirichlet* boundary conditions. How do you write the general solution to the Poisson equation for electrostatic potential $\Phi(\mathbf{x})$ using a Green's function?

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\ G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') - \frac{1}{4\pi}\int_S da'\ \frac{\partial G}{\partial n'}\ \Phi(\mathbf{x}'),$$

where the second integral is over the surface S bounding the system, and n' is a coordinate pointing out of the system, at the boundary.

Compare, for the next problem with *Neumann boundary conditions*, where the potential solution is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\ G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') + \frac{1}{4\pi}\int_S da'\ G(\mathbf{x}, \mathbf{x}')\frac{\partial \Phi}{\partial n'} + \langle \Phi \rangle_S\,.$$

The last term there is the average of the potential on the bounding surface, which is sometimes an irrelevant constant.

10. (10) In an electrostatics problem with *Neumann* boundary conditions, what is the simplest allowable boundary condition on the the Green's function $G(\mathbf{x}, \mathbf{x}')$? Hint: The result must be consistent with the differential equation that G satisfies.

The Green's function solves $\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$, and the simplest boundary condition is $\frac{\partial G}{\partial n'} = A =$ constant. The constant cannot be zero, due to the divergence theorem applied to the differential equation:

$$\int_V d^3x\ \nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi \int_V d^3x\ \delta(\mathbf{x} - \mathbf{x}') \quad \Longrightarrow \quad \oint_S \vec{\nabla} G \cdot \mathbf{n}\, da = -4\pi.$$

Then the average outward gradient of the Green's function must be

$$\left\langle \frac{\partial G}{\partial n} \right\rangle = \frac{1}{S} \oint_S \vec{\nabla} G \cdot \mathbf{n}\, da = -\frac{4\pi}{S} \quad \Longrightarrow \quad A = -\frac{4\pi}{S},$$

where S is the total surface area of the system boundary.

11. Use delta-functions to express the charge density $\rho(\mathbf{x})$ for the following charge distributions, in the indicated coordinate systems:

a) (10) A point charge q on the z-axis at $z = c$. Use cylindrical coordinates (ρ, ϕ, z).

The charge density must have $\delta(z - c)$ to get the correct location on the z-axis. Then it will also need a normalization constant A, and some dependence on radial coordinate ρ. Try

$$\rho_e(\mathbf{x}) = A f(\rho)\delta(z - c) \quad \text{and require} \int d^3x\ \rho_e(\mathbf{x}) = q.$$

Use volume element $d^3x = \rho\, d\rho\, d\phi\, dz$, then

$$\int d^3x\ \rho_e(\mathbf{x}) = A \int_0^\infty \rho f(\rho)\, d\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, \delta(z - c) = 2\pi A \int_0^\infty \rho f(\rho)\, d\rho = q.$$

The only way for this to work is if $A = q/(2\pi)$ and $f(\rho) = \delta(\rho)/\rho$. So we need

$$\rho_e(\mathbf{x}) = \frac{q}{2\pi\rho}\delta(\rho)\delta(z - c).$$

It has correct dimensions, charge $\times$ length^{-3}.

b) (10) A charge Q distributed uniformly over an infinitesimally thin circular disk of radius a centered on the z-axis and lying in the plane $z = 0$. Use spherical coordinates (r, θ, ϕ).

The charge is spread out over an area πa^2, giving a uniform surface charge density $\sigma = Q/(\pi a^2)$. We need a delta at $\theta = \pi/2$ to place the charge in the xy-plane, a constant, and some function $f(r)$ to get the uniform surface charge density. A step function keeps the charges at $r < a$. Try

$$\rho(\mathbf{x}) = A f(r)\delta\left(\theta - \frac{\pi}{2}\right) H(a - r).$$

For $r < a$, find the amount of charge in a circular ring of width dr, which should be

$$dq = \sigma\, da = \sigma 2\pi r\, dr\, H(a - r)$$

also

$$dq = r^2\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\ \rho(\mathbf{x}) = 2\pi A\ r^2\, f(r) H(a - r)$$

These require:

$$A = \sigma = \frac{Q}{\pi a^2} \quad \text{and} \quad f(r) = \frac{1}{r}.$$

Then the final charge density is

$$\rho(\mathbf{x}) = \frac{Q}{\pi a^2 r}\delta\left(\theta - \frac{\pi}{2}\right) H(a - r).$$

It is seen to have the correct dimensions.

Electrodynamics I — Midterm Exam 1 - Part B - Open Book — KSU 2015/09/25

Name Electrodynamic   Score = 130 / 130 points

Instructions: Use SI units. Please show the details of your derivations here. Explain your reasoning for full credit. Open-book only, no notes.

1. A point charge q is located a distance a above an infinite plane conductor held at zero potential. Use the method of images to find

a) (20) The surface charge density on the plane, as a function of a radial coordinate ρ.

With the charge q at $\mathbf{x}' = (0, 0, a)$, by symmetry an image $q' = -q$ at $\mathbf{x}'' = (0, 0, -a)$ will cause the total potential to be zero on the entire xy-plane at $z = 0$. As a result, the total potential at some field point $\mathbf{x} = (x, y, z)$ is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{x}-\mathbf{x}'|} - \frac{q}{|\mathbf{x}-\mathbf{x}''|}\right].$$

With $\mathbf{x} = (\rho, \phi, z)$ in cylindrical coordinates, where $\rho^2 = x^2 + y^2$, this is

$$\Phi(\rho, \phi, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\rho^2+(z-a)^2}} - \frac{1}{\sqrt{\rho^2+(z+a)^2}}\right].$$

From Gauss' Law one knows the relation, $\sigma = \epsilon_0 E_n = \epsilon_0 E_z$, which gives the surface charge density in terms of the component of electric field at the conducting surface, $z = 0$. This is

$$\sigma = \epsilon_0 E_z = -\epsilon_0 \left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = \frac{q}{4\pi}\left[\frac{z-a}{[\rho^2+(z-a)^2]^{3/2}} - \frac{z+a}{[\rho^2+(z+a)^2]^{3/2}}\right]_{z=0}$$

$$\sigma = \frac{q}{4\pi}\left[\frac{-2a}{[\rho^2+a^2]^{3/2}}\right] = \frac{-qa}{2\pi[\rho^2+a^2]^{3/2}}.$$

This diminishes away from the charge, also gets smaller if the charge is farther from the conductor, and it has the opposite sign as q.

b) (20) The force per unit area on the plane, F/A, as a function of ρ.

The forces acts on the surface charge density. In a previous question, the force per area has been found to be given from $\mathbf{F}/A = \frac{1}{2}\sigma\mathbf{E}$, using the value of the electric field at the conducting surface, which points in the $-\mathbf{z}$ direction. Because σ is also negative, the force points in the $+\mathbf{z}$ direction. The magnitude of force per area is also the same as the energy density, measured at the surface,

$$\frac{F_z}{A} = \frac{1}{2}\sigma E_z = \frac{1}{2}\epsilon_0 E_z^2 = \frac{1}{2}\epsilon_0\left(-\frac{\partial\Phi}{\partial z}\right)^2\Bigg|_{z=0} = \frac{1}{2}\epsilon_0 \times \left(\frac{q}{4\pi\epsilon_0}\frac{2a}{[\rho^2+a^2]^{3/2}}\right)^2 = \frac{(qa)^2}{8\pi^2\epsilon_0(\rho^2+a^2)^3}.$$

This can be checked to have the correct dimensions, because $q^2/(\epsilon_0\rho^2)$ is Coulomb force. The greatest concentration of force density lies directly under the charge, as expected.

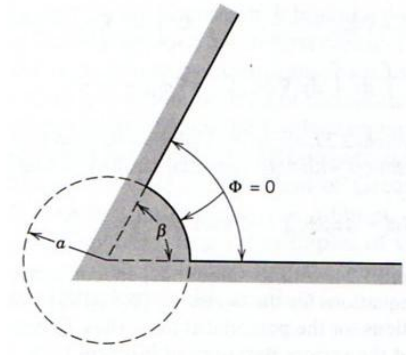
c) (20) The total electric force on the plane, by integrating the force per area. Is it the result you expect?

Using a symmetrical area element, $dA = 2\pi\rho\, d\rho$, do the integration,

$$F_z = \int \frac{F_z}{A}\, dA = \frac{(qa)^2}{8\pi^2\epsilon_0}\int_0^\infty \frac{2\pi\rho\, d\rho}{(\rho^2+a^2)^3} = \frac{(qa)^2}{8\pi\epsilon_0}\int_0^\infty \frac{d(\rho^2+a^2)}{(\rho^2+a2)^3} = \frac{(qa)^2}{8\pi\epsilon_0}\frac{1}{2a^4} = \frac{q^2}{16\pi\epsilon_0 a^2}.$$

This is the same as $F_z = q^2/(4\pi\epsilon_0 d^2)$ where $d = 2a$ is the separation of the charge an its image, so the result is the one expected.

2. A 2D region, $\rho \geq a$, $0 \leq \phi \leq \beta$ is bounded by conducting surfaces at $\phi = 0$, $\phi = 0$ and $\phi = \beta$ held at zero potential (curved 2D corner, see Fig.). The potential is determined by some charges far from the region.



a) (20) Write down a solution for the potential $\Phi(\rho, \phi)$ that satisfies the boundary conditions for finite ρ. It may have undetermined constants.

The basic solutions to 2D electrostatics as found from the Laplace equation must contain the dependencies,

$$\Phi \propto (a_\nu \rho^\nu + b_\nu \rho^{-\nu})(A_\nu e^{i\nu\phi} + B_\nu e^{-i\nu\phi})$$

with some undetermined constants. There is also a logarithmic term possible for $\nu = 0$ azimuthally symmetric part of the solution, $\Phi \propto a_0 + b_0 \ln \rho$. The total general solution looks like

$$\Phi = a_0 + b_0 \ln \rho + \sum_{\nu=1}^{\infty} (a_\nu \rho^\nu + b_\nu \rho^{-\nu})(A_\nu e^{i\nu\phi} + B_\nu e^{-i\nu\phi}).$$

One must choose constants to force $\Phi = 0$ at $\rho = a$ and also at $\phi = 0$ and $\phi = \beta$. The angular terms can be made to obey the boundary conditions using sine functions, with $\nu \to m\pi/\beta$.

$$(A_\nu e^{i\nu\phi} + B_\nu e^{-i\nu\phi}) \quad \to \quad A_m \sin\left(\frac{m\pi\phi}{\beta}\right), \quad m = 1, 2, 3, \ldots$$

Also, the radial coefficients must be chosen so each term is zeroed at $\rho \to a$,

$$\left(a_m \rho^{m\pi/\beta} + b_m \rho^{-m\pi/\beta}\right)_{\rho=a} = 0 \quad \Longrightarrow \quad b_m = -a_m\, a^{2m\pi/\beta}.$$

It is impossible to satisfy the boundary conditions at $\phi = 0,\ \beta$, if the logarithmic terms are present. Hence, one needs to chose $a_0 = b_0 = 0$. So a solution that can satisfy the BCS is

$$\Phi = \sum_{m=1}^{\infty} A_m \left(\rho^{m\pi/\beta} - a^{2m\pi/\beta} \rho^{-m\pi/\beta}\right) \sin\left(\frac{m\pi\phi}{\beta}\right).$$

b) (20) In the case that $\beta = \pi$, the problem is that of a half cylinder on an infinite plane. To leading order, determine how the surface charge density depends on ρ along the boundary at $\phi = 0, \rho > a$.

The surface charge density is again obtained from $\sigma = \epsilon_0 E_n$, in terms of the component of the electric field along the outward normal from a conducting surface. For the surface at $\phi = 0$, $\rho > a$, the outward normal is in the $\hat{\phi}$ direction. Therefore we do

$$\sigma = \epsilon_0 E_\phi = -\epsilon_0 \frac{1}{\rho} \frac{\partial \Phi}{\partial \phi}\bigg|_{\phi=0} = -\epsilon_0 \frac{1}{\rho} \sum_{m=1}^{\infty} A_m \left(\rho^{m\pi/\beta} - a^{2m\pi/\beta}\rho^{-m\pi/\beta}\right) \frac{m\pi}{\beta} \cos(0).$$

The leading dependence is the first term, $m = 1$. With $\pi/\beta = 1$, this just gives

$$\sigma \approx -\epsilon_0 \frac{1}{\rho} A_1 \left(\rho - \frac{a^2}{\rho}\right) \frac{\pi}{\beta} = -\epsilon_0 A_1 \left(1 - \frac{a^2}{\rho^2}\right).$$

This goes to zero at $\rho = a$, and nearly a constant for large ρ. Far from the half-cylinder, there will be a nearly uniform electric field perpendicular to that surface, while charge is sucked out of the corner at $\rho = a$. See Jackson problem 2.26 for more details.

3. For a point $\mathbf{x}$ in some volume V bounded by a surface S, Green's theorem can be arranged into an integral statement about electrostatic potential,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{x}')}{R} d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da' \tag{1}$$

where $R = |\mathbf{x} - \mathbf{x}'|$ and the other symbols have their usual meanings.

Consider proving the **mean-value theorem**: *For charge-free space the electrostatic potential at a point is its average over the surface of any sphere centered on that point.*

a) (5) To prove this theorem, first explain why the first term in Eq. 1 will be zero for the specified situation.

There is no charge in the considered volume, $\rho(\mathbf{x}') = 0$. That makes the first integral identically zero.

b) (10) Next, give a convincing mathematical argument showing why the second term in Eq. 1 will be zero for the specified situation.

Note that field point $\mathbf{x}$ is the center of a sphere, of radius R, whose surface is the surface of integration for the last two terms. The vector $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ points from that spherical surface, to the field point $\mathbf{x}$ at the center of the sphere.

In the first surface integral term, there is the gradient of the potential, $\partial\Phi/\partial n'$. But n' is an axis pointing out of the volume V, perpendicular to its surface. Then this gives the normal component of the electric field at the surface, $E_{n'} = \mathbf{E}\cdot\mathbf{n}' = -\partial\Phi/\partial n'$. Because R is a constant on the sphere, it comes out of the integrand. Then by Gauss' Law we get

$$\frac{1}{4\pi R}\oint_S \frac{\partial \Phi}{\partial n'}\, da' = \frac{-1}{4\pi R}\oint_S \mathbf{E}\cdot\mathbf{n}' da' = \frac{-1}{4\pi R}\frac{q_{\text{enc}}}{\epsilon_0} = 0.$$

There is no enclosed charge in the sphere's volume, by assumption, so the integral is zero.

c) (15) Finally, give a convincing mathematical argument showing what the value of the last term in Eq. (1) is, and explaining how this proves the theorem.

The gradient of $1/R$ is needed for the last term, with respect to a coordinate n' pointing outward out of the sphere volume. But that radial coordinate is the sphere's radius R. So we have

$$\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R}\left(\frac{1}{R}\right) = -\frac{1}{R^2}.$$

Again, this is a constant on the sphere, so now there results

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) da' = -\frac{1}{4\pi}\left(\frac{-1}{R^2}\right)\oint_S \Phi(\mathbf{x}')\, da' = \frac{1}{4\pi R^2}\oint_S \Phi(\mathbf{x}')\, da'$$

Of course, the integral is over the surface of the sphere, then divided by the sphere's area. That is the usual definition of the potential averaged over the surface of the sphere. So this proves the mean-value theorem. Keep in mind that it only works in a charge-free region.