Electrodynamics I   Midterm Exam - Part A - Closed Book   KSU 2014/10/23

Name Electro Dynamic

Instructions: Use SI units. Where appropriate, define all variables or symbols you use, in words. Try to tell about the physics involved, more than the mathematics, if possible.

1. (3) Write Gauss' Law in differential form. Explain the physical meaning.

$$\vec{\nabla} \cdot \vec{E}(\vec{r}\,) = \rho(\vec{r}\,)/\epsilon_0, \qquad \text{or} \qquad \vec{\nabla} \cdot \vec{D} = \rho,$$

$\vec{E}$= electric field, ρ=volume charge density, ϵ_0=permittivity of vacuum, $\vec{D}$=electric displacement. It exemplifies how electric field lines originate on positive charges and terminate on negative charges.

2. (3) Write an expression that gives the electrostatic field energy in vacuum.

$$W = \int d^3r \; \frac{1}{2}\epsilon_0 |\vec{E}|^2 = \frac{1}{2}\epsilon_0 \int d^3r \; |\vec{\nabla}\Phi|^2.$$

3. (3) Show how to get the capacitance of an isolated spherical conductor of radius R. How large in μF is the capacitance of the Earth ($R = 6380$ km), considered as a large conductor?

$$C = \frac{Q}{V} = \frac{Q}{Q/(4\pi\epsilon_0 R)} = 4\pi\epsilon_0 R.$$

For the Earth with $R = 6380 \times 10^3$ m and $\epsilon_0 = 8.854 \times 10^{-12}$ $C^2N^{-1}m^{-2}$, we get $C = 4\pi(8.854 \times 10^{-12}$ $C^2N^{-1}m^{-2} \times 6.38 \times 10^6$ m = 710 μF.

4. (3) Write a differential equation that a Green function $G(\vec{r}, \vec{r}\,')$ for Poisson's equation must satisfy, for Dirichlet boundary conditions.

$$\nabla^2 G(\vec{r}, \vec{r}\,') = -4\pi\delta(\vec{r} - \vec{r}\,').$$

$\vec{r}\,'$ is the source point, $\vec{r}$ is the observation point or field point where a response (the potential field) is measured.

5. (3) A problem has boundaries with Dirichlet boundary conditions. How do you write the solution to the Poisson equation for electrostatic potential $\Phi(\vec{r}\,)$ using a Green's function?

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d^3r' \; G(\vec{r}, \vec{r}\,')\rho(\vec{r}\,') - \frac{1}{4\pi} \int_S da\,' \frac{\partial G}{\partial n\,'} \; \Phi(\vec{r}\,').$$

where the second integral is over the surface S bounding the system, and $n\,'$ is a coordinate pointing out of the system boundary.

6. (3) Give a condition (possibly as an inequality) that identifies the limit where classical E&M theory should be replaced by quantum theory. Explain it.

Classical E&M could become invalid when the number of photons N in a volume equal to the wavelength cubed, λ^3, is much less than 1.

$$N = n\lambda^3 \ll 1$$

The number of photons per unit volume n is given from the rms energy density in the fields divided by the photon energy,

$$n = \frac{\epsilon_0 E_{\rm rms}^2}{h\nu}.$$

This will tend to happen more so at high photon energy.

7. (3) A charge density $\rho(\vec{r}\,)$ is invariant when the system is rotated through any angle around z-axis. How can you write the general solution of Poisson's equation for the potential $\Phi(\vec{r})$ in this situation?

$$\Phi(\vec{r}\,) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \rho(\vec{r}\,')\frac{1}{r_<}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma)$$

where γ is the angle between source point $\vec{r}\,'$ and field point $\vec{r}$. One still needs to do the integration over the charge density. Note that in any charge-free region, a solution of Laplace's equation is,

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty}\left[A_l r_<^l + B_l r_>^{-(l+1)}\right] P_l(\cos\theta)$$

where θ is the polar angle from z-axis to the $\vec{r}$ direction.

8. (3) A linear and isotropic dielectric medium has electric susceptibility χ. How does χ enter in the formulas for the electric polarization and the electric permittivity?

Polarization is $\vec{P} = \epsilon_0\chi\vec{E}$.
Permittivity is $\epsilon = \epsilon_0(1+\chi)$, ***from*** $\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P}$.

9. (3) Give a formula that determines the electric dipole moment of an arbitrary but localized charge density $\rho(\vec{r})$.

$$\vec{p} = \int_V d^3r \; \rho(\vec{r}\,)\;\vec{r}.$$

10. (3) If a point electric dipole $\vec{p}$ is located at position $\vec{r}_0$, what electrostatic potential does it produce at an arbitrary position $\vec{r}$?

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}.$$

where $\vec{r}-\vec{r}_0$ is the vector from the dipole to the field point.

11. (6) For the point dipole of the previous question, what electric field does it produce at an arbitrary position $\vec{r}$?

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) = \frac{3(\vec{p}\cdot\hat{n})\hat{n}-\vec{p}}{4\pi\epsilon_0|\vec{r}-\vec{r}_0|^3} - \frac{\vec{p}}{3\epsilon_0}\delta(\vec{r}-\vec{r}_0)$$

where $\hat{n} = (\vec{r}-\vec{r}_0)/|\vec{r}-\vec{r}_0|$ is the unit vector pointing from the dipole to the field point. The delta function makes the expression give the correct value of $\int d^3r\;\vec{E}$ over any volume including the dipole.

12. Use delta-functions to express the charge density $\rho(\vec{r}\,)$ for the following charge distributions, in the indicated coordinate systems:

   a) (3) A charge Q distributed uniformly over an infinitely thin circular ring of radius a centered on the z-axis and lying in the plane $z=b$. Use spherical coordinates (r,θ,ϕ).

$$\rho(\vec{r}\,) = \frac{Q}{2\pi a}\,\frac{\delta\left(r-\sqrt{a^2+b^2}\right)}{\sqrt{a^2+b^2}}\delta\left(\theta - \sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right).$$

   Once this is integrated over all space with $d^3r = r^2\,dr\,\sin\theta\,d\theta$ the total charge Q will be recovered.

b) (3) A point charge q on the x-axis at $x = x_0$. Use cylindrical coordinates (ρ, ϕ, z).

$$\rho(\vec{r}\,) = q\,\frac{\delta\,(\rho - x_0)}{\rho}\,\delta(\phi)\,\delta(z).$$

Once this is integrated over all space with $d^3r = \rho\,d\rho\,d\phi\,dz$ the charge a will be recovered.

13. A point charge q is placed at a distance $d > a$ from the center of an *uncharged* isolated metal sphere of radius a.

a) (6) Determine the electric force acting on q due to the sphere, for arbitrary $d > a$. Is it attractive or repulsive? Explain.

This can be solved either by an image within the sphere, or by using the azimuthal symmetry of the situation. Using azimuthal symmetry, the solution for potential outside the sphere can be written as a term directly due to the charge at $r = d$ and other terms due to the induced charges on the sphere.

$$\Phi(\vec{r}\,) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x} - d\hat{z}|} + \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta)\right].$$

The summation is due to to induced surfaces charges on the sphere. For Φ to be finite at large r, all $A_l = 0$. The term coming directly from the charge can also be expanded in Legendre polynomials, giving

$$\Phi(\vec{r}\,) = \frac{1}{4\pi\epsilon_0}\left[q\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) + \sum_{l=0}^{\infty}B_l r^{-(l+1)}P_l(\cos\theta)\right].$$

where $r_<, r_>$ are the min and max of r and d. Now the potential on the surface sphere must be a constant (not necessarily zero, the sphere is uncharged, not grounded). However, setting the term $B_0 = 0$ will correspond to an uncharged sphere–this is the monopole charge of the sphere! Now at $r = a$, all other factors for $l > 0$ must give zero, with $r_< = r = a, r_> = d$:

$$q\frac{a^l}{d^{l+1}} + \frac{B_l}{a^{l+1}} = 0, \quad l > 0$$

which is solved by

$$B_l = -q\frac{a^{2l+1}}{d^{l+1}} = \left(-q\frac{a}{d}\right)\left(\frac{a^2}{d}\right)^l = q'(d')^l.$$

That shows essentially an image $q' = -q\frac{a}{d}$ at position $d' = \frac{a^2}{d}$ within the sphere. By making $B_0 = 0$ also, is equivalent to an additional image $q'' = -q' = +q\frac{a}{d}$ at the center of the sphere.

To get the force, first find the electric field only due to the B_l terms. This avoids self-interaction on q. The z-component of the field is all that is needed, which will come from E_r with $\theta = 0$ and $P_l(\cos(0)) = P_l(1) = 1$.

$$E_r = -\frac{\partial\Phi}{\partial r} = \frac{-1}{4\pi\epsilon_0}\sum_{l=1}^{\infty}-(l+1)B_l r^{-(l+2)}P_l(\cos\theta)$$

$$E_z = \frac{q'}{4\pi\epsilon_0}\sum_{l=1}^{\infty}(l+1)\frac{(d')^l}{r^{l+2}}\cdot 1 = \frac{q'}{4\pi\epsilon_0}\left[\frac{1}{(r-d')^2} - \frac{1}{r^2}\right]$$

The last step comes from looking at

$$\frac{1}{|r_> - r_<|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}, \qquad \frac{\partial}{\partial r_<}\frac{1}{|r_> - r_<|} = \frac{1}{|r_> - r_<|^2} = \sum_{l=0}^{\infty}(l+1)\frac{r_<^l}{r_>^{l+2}}$$

and shifting out the $l = 0$ term. Finally the general result for the force is (with $r = d$)

$$F = qE_z = \frac{qq'}{4\pi\epsilon_0}\left[\frac{1}{(d-d')^2} - \frac{1}{d^2}\right] = -\frac{q^2a/d}{4\pi\epsilon_0}\left[\frac{1}{(d-a^2/d)^2} - \frac{1}{d^2}\right].$$

Negative value shows that the force is attractive for any d.

b) (4) Find the asymptotic force law for $d \gg a$.

Try to combine the terms in the brackets in the last force expression. The first factor can be expanded,

$$\frac{1}{(d-a^2/d)^2} = \frac{1}{d^2}\frac{1}{(1-a^2/d^2)^2} \approx \frac{1}{d^2}\left(1 + 2\frac{a^2}{d^2}\right)$$

Then combining with the other term, this gives

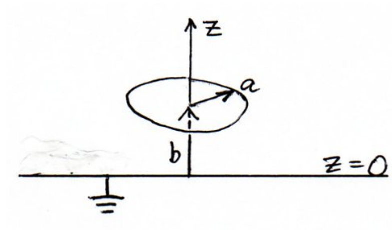
$$F \approx -\frac{q^2a/d}{4\pi\epsilon_0}\left[\frac{1}{d^2}\left(1+2\frac{a^2}{d^2}\right) - \frac{1}{d^2}\right] = -\frac{2q^2a^3}{4\pi\epsilon_0 d^5}.$$

A very weak attractive force, even with an uncharged sphere! This is due to the polarization induced in the sphere by the source charge q.

Name____

Instructions: Use SI units. Please show the details of your derivations here. Explain your reasoning for full credit. Open-book only, no notes.

1. An infinitely thin ring of total charge Q has a radius a, and is placed centered on the z-axis in the plane $z = b$, above a *grounded* infinite plane conductor at $z = 0$. The plane of the ring is parallel to the plane of the conductor.



a) (8) Find the electric potential $\Phi(z)$ along the z-axis anywhere $z > 0$, which is the axis of the ring.

The grounded infinite plane can be modeled by putting an image ring of charge $-Q$ centered on the z-axis but in the plane $z = -b$, below the xy-plane (the mirror image of the original ring). Any point on the z-axis is equidistant from all of the charge in either the original ring or the image ring. At position z, it is then simple to write down the total potential,

$$\Phi(z) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{\sqrt{a^2+(z-b)^2}} + \frac{-Q}{\sqrt{a^2+(z+b)^2}}\right)$$

b) (8) Expand your result of part a in power series, one valid for $z < \sqrt{a^2+b^2}$, and another series valid for $z > \sqrt{a^2+b^2}$.

The direct term from the original ring depends on

$$\frac{1}{\sqrt{a^2+(z-b)^2}} = \frac{1}{\sqrt{a^2+b^2+z^2-2bz}} = \frac{1}{\sqrt{c^2+z^2-2cz\cos\alpha}}$$

where one has Pythagorean relation $c^2 = a^2 + b^2$ and angle definition $\cos\alpha = b/c$. That is, α is the angle between the z-axis and a radius from the origin out to the ring. But this is of the form of a reciprocal of a difference of c and z:

$$\frac{1}{\sqrt{a^2+(z-b)^2}} = \frac{1}{|\vec{c} - z\hat{z}|}$$

where $\vec{c}$ is of length c and polar angle direction $\theta = \alpha$ (with arbitrary azimuthal direction, for instance, let it have $\phi = 0$). Then we know it expansion from knowledge of Legendre polynomials,

$$\frac{1}{|z\hat{z} - \vec{c}|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha)$$

and $r_<, r_>$ are the min, max of z and $c = \sqrt{a^2+b^2}$.

For the image ring, we need the expansion with the opposite sign on b,

$$\frac{1}{\sqrt{a^2+(z+b)^2}} = \frac{1}{\sqrt{a^2+b^2+z^2+2bz}} = \frac{1}{\sqrt{c^2+z^2+2cz\cos\alpha}}$$

Due to the opposite sign on the cosine term, it can also be expanded the same way, but it is equivalent to shifting α by π (the angle from $+z$-axis to the image ring is $\alpha+\pi$). Then the expansion needed for the image is

$$\frac{1}{|z\hat{z}+\vec{c}|}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos(\alpha+\pi))$$

The shift on α reverses the sign of the argument in P_l, i.e., $\cos(\alpha+\pi)=-\cos\alpha$. The the total potential on the z-axis is

$$\Phi(z)=\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|z\hat{z}-\vec{c}|}-\frac{1}{|z\hat{z}+\vec{c}|}\right)=\frac{Q}{4\pi\epsilon_0}\sum_{l=1,3,5\dots}^{\infty}\frac{r_<^l}{r_>^{l+1}}\,2P_l(\cos\alpha)$$

Combining the two contributions involves the combinations

$$P_l(\cos\alpha)-P_l(-\cos\alpha)=[1-(-1)^l]P_l(\cos\alpha)=$$

which relies on the fact that P_l with odd l are odd functions, resulting in only a sum over the odd Legendre polynomials.

Summarizing, for $z<c=\sqrt{a^2+b^2}$ we have

$$\Phi(z)=\frac{2Q}{4\pi\epsilon_0}\sum_{l=1,3,5\dots}^{\infty}\frac{z^l}{c^{l+1}}P_l(\cos\alpha)=\frac{2Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{z^{2l+1}}{c^{2l+2}}P_{2l+1}(\cos\alpha)$$

and for $z>c$ we have

$$\Phi(z)=\frac{2Q}{4\pi\epsilon_0}\sum_{l=1,3,5\dots}^{\infty}\frac{c^l}{z^{l+1}}P_l(\cos\alpha)=\frac{2Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{c^{2l+1}}{z^{2l+2}}P_{2l+1}(\cos\alpha)$$

c) (8) Use the result of part b to find the electric potential $\Phi(r,\theta,\phi)$ for any points above the grounded plane.

Once the potential is known on the z-axis, points off of that axis are found by assuming the general form for azimuthal symmetry,

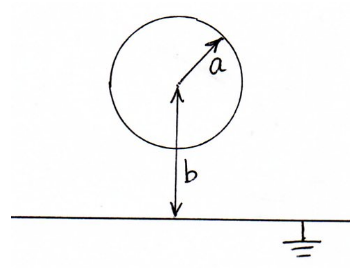
$$\Phi(\vec{r})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[A_l r^l+B_l r^{-(l+1)}\right]P_l(\cos\theta)$$

To match to the solution on the z-axis, use $P_l(\cos(0))=1$, then the expansion coefficients can be read off of the solution for $\Phi(z)$. The net result is that once the different regions $z<c$ and $z>c$ are considered, one needs to simply include the factor of $P_l(\cos\theta)$ into the solution for $\Phi(z)$ to extend it away from the z-axis:

$$\Phi(r,\theta)=\frac{2Q}{4\pi\epsilon_0}\sum_{l=1,3,5\dots}^{\infty}\frac{r_<^l}{r_>^{l+1}}\,P_l(\cos\alpha)P_l(\cos\theta)=\frac{2Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^{2l+1}}{r_>^{2l+2}}\,P_l(\cos\alpha)P_l(\cos\theta)$$

where again $r_<,r_>$ are the min and max of r and $c=\sqrt{a^2+b^2}$. One can then also express it separately in the two regions $z<c$ and $z>c$.

2. A very long conducting cylinder with a circular cross section of radius a is placed with its axis a distance $b > a$ away from and parallel to a grounded plane conductor. The cylinder is held at fixed potential V relative to the grounded plane.



a) (10) Use the method of images and show that by an appropriate choice of image line charges, the equipotentials are circles. Hint: The image line charge within the cylinder does not need to be along its axis.

This is a 2D geometry (x along the plane, y perpendicular), and we know from different homework problems that we can place the original line charge within the cylinder (but not at its center), and an oppositely charged image (line) charge below the plane a to cause the cylinder and plane to be equipotentials. Place these line charges on the y-axis at $y = \pm c$, where constant c is to be determined. By such placement, already the plane is grounded ($\Phi = 0$ there). Each line of charge with linear charge density λ makes an outward electric field

$$E_\rho = \frac{\lambda}{2\pi\epsilon_0 \rho}$$

and the corresponding contribution to potential, for one line of charge, is,

$$\Phi(\rho) = -\frac{\lambda}{2\pi\epsilon_0} \ln \rho$$

where an arbitrary reference potential has been dropped. Then the total (2D) potential at positions above the plane but outside the cylinder will be

$$\Phi(x,y) = -\frac{\lambda}{2\pi\epsilon_0} \ln \sqrt{x^2 + (y-c)^2} + \frac{\lambda}{2\pi\epsilon_0} \ln \sqrt{x^2 + (y+c)^2}$$

$$\Phi(x,y) = \frac{\lambda}{4\pi\epsilon_0} \ln \left(\frac{x^2 + (y+c)^2}{x^2 + (y-c)^2} \right)$$

By design, on $y = 0$, we have $\Phi(x,0) = 0$ for the grounded plane.

For other equipotentials at $\Phi = V$, some other constant, re-arrange to get their depedence on (x,y).

$$\exp(4\pi\epsilon_0 V/\lambda) \left[x^2 + (y-c)^2 \right] = x^2 + (y+c)^2$$

$$\left(e^{4\pi\epsilon_0 V/\lambda} - 1 \right) (x^2 + y^2 + c^2) - 2cy \left(e^{4\pi\epsilon_0 V/\lambda} + 1 \right) = 0$$

$$x^2 + y^2 - 2cy \coth \left(\frac{2\pi\epsilon_0 V}{\lambda} \right) + c^2 = 0$$

Already that is quadratic in x and y with equal factors on x^2 and y^2, hence the equipotentials are circles.

b) (8) Find how the circle center and radius of an equipotential circle depend on a chosen value of potential Φ between 0 and V.

Re-arange some more the equation for the equipotentials,

$$x^2 + \left[y - c\coth\left(\frac{2\pi\epsilon_0 V}{\lambda}\right)\right]^2 = c^2\left[\coth^2\left(\frac{2\pi\epsilon_0 V}{\lambda}\right) - 1\right]$$

$$x^2 + \left[y - c\coth\left(\frac{2\pi\epsilon_0 V}{\lambda}\right)\right]^2 = c^2\,\mathrm{csch}^2\left(\frac{2\pi\epsilon_0 V}{\lambda}\right)$$

From this we can read off the radius R of the circle and its center y_0,

$$R = c\,\mathrm{csch}\left(\frac{2\pi\epsilon_0 V}{\lambda}\right)$$

$$y_0 = c\coth\left(\frac{2\pi\epsilon_0 V}{\lambda}\right)$$

such that the equation for the circular equipotentials is the standard form,

$$x^2 + (y - y_0)^2 = R^2.$$

c) (6) Calculate the capacitance per unit length of the cylinder/plane system.

We take $R = a$ which is the size of the cylindrical conductor, and $y_0 = b$ which is the distance of its center from the grounded plane. Then these combine to give

$$R = a = c\frac{1}{\sinh(2\pi\epsilon_0 V/\lambda)}, \quad y_0 = b = c\frac{\cosh(2\pi\epsilon_0 V/\lambda)}{\sinh(2\pi\epsilon_0 V/\lambda)}$$

hence

$$\frac{b}{a} = \cosh\frac{2\pi\epsilon_0 V}{\lambda}.$$

Then the capacitance per unit length is the charge per unit length divided by the potential,

$$\frac{C}{L} = \frac{\lambda}{V} = \frac{2\pi\epsilon_0}{\cosh^{-1}(b/a)}$$

As $\epsilon_0 = 8.854$ pF/m, one can see that this has the correct dimensions. Largest values will result when b is nearly equal to a, that is, the cylinder is as close as possible to the plane.

d) (6) Bonus. Find the charge density induced on either the cylinder or on the plane, as a function of angular or linear coordinate on each, respectively. (Do only one or the other.)

For the surface charge density on the grounded plane, use the basic formula:

$$\sigma = \epsilon_0 E_y = -\epsilon_0 \left.\frac{\partial\Phi}{\partial y}\right|_{y=0}$$

Then apply it to the potential found:

$$\sigma = -\epsilon_0\frac{\lambda}{4\pi\epsilon_0}\frac{\partial}{\partial y}\ln\left(\frac{x^2 + (y+c)^2}{x^2 + (y-c)^2}\right) =$$

$$= -\frac{\lambda}{4\pi}\left[\frac{2(y+c)}{x^2 + (y+c)^2} - \frac{2(y-c)}{x^2 + (y-c)^2}\right]_{y=0} = -\frac{\lambda c/\pi}{x^2 + c^2}$$

But now the definitions of a and b lead to a simple result,

$$b^2 - a^2 = c^2 \left(\coth^2 \frac{2\pi\epsilon_0 V}{\lambda} - \operatorname{csch}^2 \frac{2\pi\epsilon_0 V}{\lambda} \right) = c^2.$$

Then the surface density on the plane, in terms of the cylinder parameters, is

$$\sigma_{\text{plane}} = \left(-\frac{\lambda}{\pi} \right) \frac{\sqrt{b^2 - a^2}}{x^2 + b^2 - a^2}.$$

As the dimensions of λ are charge/length, this has the correct dimensions of charge/area. Further, it will be of greatest magnitude at $x = 0$, at the closest point between the cylinder and the plane. It has the opposite sign as the charge on the cylinder.

For the surface density on the cylinder, use:

$$\sigma = \epsilon_0 E_r = -\epsilon_0 \left. \frac{\partial \Phi}{\partial r} \right|_{r=a}$$

where r is the radius measured from the center of the cylinder. Use the coordinates like this:

$$x = r \sin\theta, \quad y = b - r\cos\theta.$$

where $\theta = 0$ gives the point on the cylinder closest to the plane. Then in these coordinates the potential is

$$\Phi = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{r^2 \sin^2\theta + (b + c - r\cos\theta)^2}{r^2 \sin^2\theta + (b - c - r\cos\theta)^2} \right]$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{r^2 + (b+c)^2 - 2(b+c)r\cos\theta}{r^2 + (b-c)^2 - 2(b-c)r\cos\theta} \right]$$

Take the derivatives,

$$\sigma = \frac{-\lambda}{4\pi} \left[\frac{2r - 2(b+c)\cos\theta}{r^2 + (b+c)^2 - 2(b+c)r\cos\theta} - \frac{2r - 2(b-c)\cos\theta}{r^2 + (b-c)^2 - 2(b-c)r\cos\theta} \right]_{r=a}$$

$$= \frac{-\lambda}{4\pi} \left[\frac{a - (b+c)\cos\theta}{b^2 + bc - a(b+c)\cos\theta} - \frac{a - (b-c)\cos\theta}{b^2 - bc - a(b-c)\cos\theta} \right]$$

$$= \frac{-\lambda}{4\pi} \left[\frac{\frac{a}{b+c} - \cos\theta}{b - a\cos\theta} - \frac{\frac{a}{b-c} - \cos\theta}{b - a\cos\theta} \right] = \frac{-\lambda}{4\pi} \left(-\frac{2c}{a} \right) \frac{1}{b - a\cos\theta}$$

Finally summarizing in terms of the cylinder parameters,

$$\sigma_{\text{cylinder}} = \left(\frac{\lambda}{2\pi a} \right) \frac{\sqrt{b^2 - a^2}}{b - a\cos\theta}.$$

This peaks at $\theta = 0$, which is the point on the cylinder closest to the plane, as can be expected. It also has a positive sign as necessary for a cylinder at positive potential.