Name____Electro Dynamic____

Instructions: Use SI units. Please make your answers brief and clear.

1. (2) Give an expression for the force $d\vec{F}$ on a current element $i\,d\vec{l}$ in a magnetic induction $\vec{B}$.

$$d\vec{F} = i\,d\vec{l} \times \vec{B}$$

which is the vector cross product of the current element with the applied magnetic induction.

2. (2) A distribution of current $\vec{J}(\mathbf{r})$ exists in a region of magnetic induction $\vec{B}(\mathbf{r})$. Write an expression for the total torque on the current distribution.

$$\vec{\tau} = \int \mathbf{r} \times d\vec{F} = \int \mathbf{r} \times \left[\vec{J}(\mathbf{r})d^3r \times \vec{B}(\mathbf{r})\right] = \int d^3r\ \vec{\mathbf{r}} \times \left[\vec{J}(\mathbf{r}) \times \vec{B}(\mathbf{r})\right]$$

3. (2) For a current distribution $\vec{J}(\mathbf{r})$, how does one express the vector potential $\vec{A}(\mathbf{r})$ that it produces?

$$\vec{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\ \frac{\vec{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

where μ_0 is the permeability of free space.

4. (2) A flat coil (loops confined in some plane) has N turns of some arbitrary two-dimensional shape (along some path $r(\theta)$, for example). If it carries a current i, how do you write an expression for its magnetic dipole moment $\vec{m}$?

$$\vec{m} = \frac{1}{2}\oint \vec{\mathbf{r}} \times Ni\,d\vec{l}(\mathbf{r}) = NiA\hat{z}$$

where $d\vec{l}(\mathbf{r})$ describes the path of the loops, and A is the enclosed area whose normal points in the $\hat{z}$ direction.

5. (2) A particle of charge q and mass M makes a periodic orbital motion. Write an expression relating its orbital magnetic moment $\vec{m}$ and its orbital angular momentum $\vec{L}$.

$$m = \frac{qA}{T} = \frac{q\pi r^2}{T} = \frac{qr}{2}\frac{2\pi r}{T} = \frac{qMvr}{2M} \quad \text{or} \quad \vec{m} = \frac{q}{2M}\vec{L}.$$

6. (2) What is the physical definition of magnetization $\vec{M}(\mathbf{r})$?

Magnetization $\vec{M}(\mathbf{r})$ is the local value of average magnetic moment per unit volume. It means that by averaging over some small but macroscopic volume ΔV centered at $\mathbf{r}$, one defines it as

$$\vec{M}(\mathbf{r}) = \frac{1}{\Delta V}\sum_{i\in\Delta V}\vec{m}_i$$

where the sum is over all magnetic dipoles $\vec{m}_i$ within ΔV.

7. (2) Give the constitutive relation between magnetic induction, magnetization, and magnetic field.

$$\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right).$$

8. (4) Use delta-functions in spherical coordinates to express the current density $\vec{J}(\mathrm{r})$ for the following situation: A spherical shell of radius a, with surface charge density $\sigma = \sigma_0 \sin\theta$, that is rotating around the z-axis at angular frequency ω.

Use the motion of the surface charge density as $\vec{v} = \omega\hat{z} \times \vec{r} = \omega r\hat{z} \times \hat{r} = \omega r\hat{\phi}$, and then write

$$\vec{J}(\mathrm{r}) = \rho\vec{v} = \sigma\delta(r-a)\vec{v} = \sigma_0 \sin\theta\delta(r-a)\omega r\hat{\phi} = \sigma_0\omega a \sin\theta\ \delta(r-a)\ \hat{\phi}.$$

9. (2) A square $a \times a$ coil has N turns. A uniform magnetic induction $B = B_0 \sin(\omega t)$ passes perpendicularly through the coil. Calculate the time-dependent emf produced in the coil.

$$\epsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int N\vec{B}\cdot d\vec{A} = -\frac{d}{dt}NB_0 a^2 \sin(\omega t) = -NB_0 a^2 \omega \cos(\omega t).$$

10. (2) Give an expression for the time-dependent electric field in terms of scalar and vector potentials.

$$\vec{E}(\mathrm{r}, t) = -\vec{\nabla}\Phi - \frac{\partial}{\partial t}\vec{A}$$

11. (4) In the presence of any time-dependent sources, what equation is obeyed by the vector potential, when using the Lorentz gauge?

It is a wave equation, driven by the current density $\vec{J}(\mathrm{r}, t)$ scaled by μ_0,

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\mu_0\vec{J}(\mathrm{r}, t)$$

12. (4) Consider a wave equation, $\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi = -4\pi f(\mathbf{r}, t)$, where $f(\mathbf{r}, t)$ is the source that drives some waves $\Psi(\mathbf{r}, t)$. Write out the space- and time-dependent Green's function for this equation that applies to a problem where the source turns on at time $t = 0$.

$$G(\mathrm{r}, t; \mathrm{r}', t') = \frac{1}{|\mathrm{r}-\mathrm{r}'|}\delta\left(t' - t + \frac{|\mathrm{r}-\mathrm{r}'|}{c}\right)$$

This produces a response at time t from a source at time t', related by

$$t = t' + \frac{|\mathrm{r}-\mathrm{r}'|}{c}$$

that is, the response comes after the propagation time for the waves to arrive at the observer's position.

13. (4) Write out an equation for Poynting's theorem in differential form. Explain in words what each term means physically.

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} + \vec{J}\cdot\vec{E} = 0.$$

where u is EM energy density, $\vec{S} = \vec{E}\times\vec{H}$ is the Poynting vector, and $\vec{J}$ and $\vec{E}$ are current density and electric field. The first term is the increase in EM field energy in a volume element. The second term is the flux of EM energy into that volume element. The third term is the mechanical work done on charges in that volume by the electric field. Poynting's theorem demonstrates that total mechanical plus EM energy is conserved.

14. (4) From consideration of Maxwell's equations, what are the symmetry properties (odd or even) of the electric polarization $\vec{P}$ under space inversion? What about time inversion?

From the relation $\vec{D} = \epsilon_0\vec{E} + \vec{P}$, the transformation properties of $\vec{P}$ are the same as those for $\vec{E}$. From Gauss' Law,

$$\vec{\nabla} \cdot \vec{E} = -\rho/\epsilon_0,$$

with ρ being scalar (even under space inversion), then $\vec{E}$ and hence $\vec{P}$ are odd under space inversion (they are vectors). Time does not appear in this one of Maxwell's equations. Therefore, $\vec{E}$ and $\vec{P}$ are even under time reversal.

15. (4) A plane EM wave is traveling in the z-direction in a medium with $\mu = \mu_0$ and $\epsilon = 4\epsilon_0$. With linearly polarized $\vec{E} = E_0\hat{x}\ \exp[i(kz - \omega t)]$ write an expression for $\vec{B}(z, t)$ in this wave.

Faraday's Law gives the relation between $\vec{E}$ and $\vec{B}$, for harmonic time dependence ($\partial/\partial t \longrightarrow -i\omega$), and with $\vec{\nabla} \longrightarrow i\vec{k}$, and dispersion relation $k = \omega\sqrt{\epsilon\mu}$,

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = i\vec{k} \times \vec{E} - i\omega\vec{B} = 0 \quad \text{or} \quad \vec{B} = \frac{1}{\omega}\vec{k} \times \vec{E} = \sqrt{\epsilon\mu}\,\hat{n} \times \vec{E}$$

where $\hat{n} = \vec{k}/k = \hat{z}$. The cross product $\hat{z} \times \vec{E} \sim \hat{y}$, and $\sqrt{\epsilon\mu} = 2\sqrt{\epsilon_0\mu_0} = 2/c$. We get

$$\vec{B} = \frac{2}{c}\, E_0\, \hat{y}\ \exp[i(kz - \omega t)].$$

16. (4) A plane wave travels in the x-direction: $\vec{E}(\mathbf{r}, t) = E_0(\hat{y} - i\hat{z}) \exp[i(kx - \omega t)]$. Looking into the wave at a fixed point in space, in which direction does the electric field vector rotate (clockwise or couterclockwise)? Which circular polarization is this (right or left)?

Taking a fixed space point ($x = 0$), and looking at the time-dependence, use the implied real part to get the components of $\vec{E}$:

$$\vec{E} = \mathrm{Re}\,\{E_0(\hat{y} - i\hat{z})\exp(-i\omega t)\} = E_0\,[\hat{y}\cos(\omega t) - \hat{z}\sin(\omega t)]$$

For small times $t > 0$ the $\hat{z}$ component increases negatively, and the rotation of the wave towards the observer is <u>clockwise</u>. Pointing your right thumb back towards the source, your right 4 fingers rotate in the clockwise sense. This is <u>right circular</u> polarization.

17. (2) Write an expression for the dielectric function $\epsilon(\omega)$ in a plasma.

$$\epsilon(\omega) = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

where the squared plasma frequency ω_p depends on the total volume density of free charges n, according to

$$\omega_p^2 = \frac{ne^2}{m\epsilon_0}$$

and m is the carrier mass (usually, the electron mass).

18. (2) What does $\epsilon(\omega)$ imply for EM waves of low frequency traveling in a plasma?

If $\omega < \omega_p$, then the dielectric function becomes negative, $\epsilon < 0$. The wave vector $k = \omega\sqrt{\epsilon\mu}$ becomes pure imaginary, leading to strong damping of the waves entering a plasma, over a short distance. Waves with frequencies below ω_p do not propagate through a plasma.

Name______________________________________

Instructions: Use SI units. Please write your derivations and final answers on these pages. Explain your reasoning for full credit. Discuss the physics if the math is impossible.

1. (18) Consider a straight wire of radius a and length l in direction z, with a current density $\vec{J} = \sigma\vec{E}$ that is uniform over its cross section. $\vec{E}$ outside the wire is assumed to be negligible here.

   a) (6) For a DC current through the wire, find the magnetic induction $\vec{B}$ both inside and outside the wire.

   Suppose the total current is $I = \pi a^2 J$. Outside the wire, application of Ampere's Law is trivial and goes like:

   $$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}} \quad\Longrightarrow\quad B_\phi = \frac{\mu_0 I}{2\pi\rho} = \frac{\mu_0 \pi a^2 J}{2\pi\rho} = \frac{\mu_0 a^2 J}{2\rho} = \frac{\mu_0 a^2 \sigma E}{2\rho}, \quad \rho > a.$$

   For the region inside the wire, the enclosed current out to radius $\rho < a$ will be instead $I_{\text{enc}} = I\rho^2/a^2$, then the magnetic induction is obtained by

   $$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}} \quad\Longrightarrow\quad B_\phi = \frac{\mu_0 I\rho^2/a^2}{2\pi\rho} = \frac{\mu_0 I\rho}{2\pi a^2} = \frac{\mu_0 J\rho}{2} = \frac{\mu_0 \sigma E\rho}{2}, \quad \rho < a.$$

   b) (6) Based on your result for the DC magnetic field, determine the Poynting vector $\vec{S}$ inside the wire, as a function of the radial coordinate ρ from the axis of the wire.

   Here evaluate Poynting vector $\vec{S} = \vec{E}\times\vec{H}$, using the magnetic field inside the wire $\vec{H} = \vec{B}/\mu_0$, and uniform electric field $\vec{E} = \vec{J}/\sigma$.

   $$\vec{S} = \vec{E}\times\vec{H} = \frac{J\hat{z}}{\sigma}\times\frac{1}{\mu_0}\frac{\mu_0 J\rho\hat{\phi}}{2} = \frac{J^2\rho}{2\sigma}(\hat{z}\times\hat{\phi}) = -\frac{J^2\rho}{2\sigma}\hat{\rho} = -\frac{\sigma E^2\rho}{2}\hat{\rho}.$$

   This is a flow of EM energy towards the axis of the wire!

   c) (6) Show that the result for $\vec{S}$ satisfies Poynting's theorem applied to the whole volume of the wire segment. Comment on the physical significance of the terms in the equation.

   Integral form for Poynting's theorem is

   $$\int_V d^3r\, \left[\frac{\partial u}{\partial t} + \vec{J}\cdot\vec{E} + \vec{\nabla}\cdot\vec{S}\right] = 0$$

   The first term is the increase in EM energy in the wire (which is zero in steady state), the second term in the work done on free charges, and the last is the flux of EM energy out of the volume. The work done on the charges (becomes heat) is

   $$\int d^3r\, \vec{J}\cdot\vec{E} = \int_0^{2\pi} d\phi \int_0^l dz \int_0^a \rho d\rho\, \frac{J^2}{\sigma} = \frac{(\pi a^2 l)J^2}{\sigma} = (\pi a^2 l)\sigma E^2.$$

   The flux of $\vec{S}$ is changed to a surface integral at the curved outer surface, $\rho = a$, $\hat{n} = \hat{\rho}$,

   $$\int d^3r\, \vec{\nabla}\cdot\vec{S} = \int \vec{S}\cdot\hat{n}\, dA = -\frac{J^2 a}{2\sigma}2\pi a l = -\frac{(\pi a^2 l)J^2}{\sigma} = -(\pi a^2 l)\sigma E^2.$$

   The negative sign shows that it is actually a flux of EM energy <u>into</u> the wire. Thus, Poynting's theorem is satisfied. This flux of energy becomes the work done on the free charges and accounts for the ohmic heating of the wire. [Also note: Can show $\vec{\nabla}\cdot\vec{S} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho S_\rho) = -\sigma E^2$ and $\vec{J}\cdot\vec{E} = \sigma E^2$, which sum to zero. These are local expressions of the terms in Poynting's theorem.]

2. (18) Consider again a straight wire as in the previous question, with $\vec{J} = \sigma \vec{E}$.

a) (6) Now suppose the current is driven through the wire at a high frequency ω, i.e, harmonic fields varying as $\exp(-i\omega t)$. Apply Maxwell's equations to such a situation to get the differential equation that E_z should solve inside the wire.

Write out Ampere's Law and combine with Faraday's Law and $\vec{J} = \sigma \vec{E}$.

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}, \quad \vec{\nabla} \times \vec{H} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial}{\partial t} \vec{\nabla} \times \vec{H} = -\mu_0 \frac{\partial}{\partial t} \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = -\mu_0 \left(\sigma \frac{\partial \vec{E}}{\partial t} + \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

With $\vec{\nabla} \cdot \vec{E} = 0$ (no net free charge), and using $-i\omega$ for time derivatives, this is

$$-\nabla^2 \vec{E} = \left(i\omega\mu_0\sigma + \omega^2 \mu_0 \epsilon_0 \right) \vec{E} \equiv k^2 \vec{E}$$

This applies to any component of $\vec{E}$, including E_z. If E_z depends on z as $e^{ik_z z}$ and on the radial coordinate ρ, then

$$\nabla^2 E_z = -k_z^2 E_z + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial E_z}{\partial \rho} \right) = -k_z^2 E_z + \frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} = -k^2 E_z,$$

This is Bessel's equation for $J_0(k_\rho \rho)$, with a complex wave vector $k_\rho = \sqrt{k^2 - k_z^2}$.

b) (6) Consider the case of a very good conductor. Explain physically why you should expect the electric field (and $\vec{J}$) to be nonuniform now within the wire. How should $|\vec{E}|$ vary with ρ inside the wire? I am looking for a very approximate answer; it may not require a calculation.

If the conductivity is very high, then for some range of frequencies one has $k^2 \approx i\omega\mu_0\sigma$. Taking the square root to get k, we have $\sqrt{i} = \sqrt{e^{i\pi/2}} = e^{i\pi/4} = (1+i)/\sqrt{2}$, then

$$k \approx (1+i)\sqrt{\frac{\omega\mu_0\sigma}{2}} = \frac{1+i}{\delta}$$

where $\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}$ is the skin depth. The electric and magnetic fields will be concentrated at the surface of the wire, causing E_z that will tend to increase with ρ. Higher frequency fields will have difficulty to penetrate much below the skin depth. A complete mathematical solution corresponds to a waveguide problem and should show this more rigorously.

c) (6) For a copper wire of radius $a = 1.00$ mm, $\epsilon = \epsilon_0$ and $\mu = \mu_0$ and $\sigma = 5.95 \times 10^7$ $(\Omega \cdot \text{m})^{-1}$, estimate the angular frequency ω above which one needs to account for this spatial variation of the fields inside the wire.

The spatial variation of the fields will be important when the skin depth is similar in size to the radius of the wire. The skin depth is seen in the above derivations. Then put

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \approx a$$

and solving for the angular frequency,

$$\omega \approx \frac{2}{\mu_0\sigma a^2} = \frac{2}{(4\pi \times 10^{-7})(5.95 \times 10^7)(10^{-3})^2} = 27000 \text{ s}^{-1}$$

This is a surprisingly low frequency above which the current density will not be uniform inside the wire! If the wire is thinner, say $a = 0.100$ mm, this frequency limit is $100\times$ higher, because there is much less interior volume into which fields could penetrate.

3. (16) Hydrogen gas (density = 10^{12} H_2 molecules per cm^3) is heated to a very high temperature ($k_B T \gg 13.6$ eV) so that all molecules are broken apart and the atoms are ionized.

a) (8) Estimate the range of angular frequencies ω of propagating EM waves in the plasma. Give a numerical result.

EM waves will propagate in the plasma only if their angular frequency is higher than the plasma frequency, ω_p, given by

$$\omega_p^2 = \frac{ne^2}{m\epsilon_0}$$

where n is the volume density of free charges and m is their mass. The only free charges here are the ionized electrons. One has an electron from every H-atom. Therefore, $n = 2.0 \times 10^{12}/cm^3 = 2.0 \times 10^{18}/m^3$. This gives the plasma frequency,

$$\omega_p = \sqrt{\frac{(2.0 \times 10^{18})(1.602 \times 10^{-19})^2}{(9.11 \times 10^{-31})(8.854 \times 10^{-12})}} = 8.0 \times 10^{10}\ \mathrm{s}^{-1}$$

This corresponds to frequency $f_p = \omega_p/2\pi \approx 13$ GHz, so typical radio waves and microwaves will not pass through this plasma.

b) (8) Suppose a plane EM wave of amplitude E_0 originally traveling in vacuum is incident on this plasma at normal incidence. The wave has a frequency $\omega = \omega_p/\sqrt{2}$, where ω_p is the plasma frequency. Use the Fresnel formulas to find the amplitude of the electric field after it travels a distance of 5λ into the plasma, where $\lambda = 2\pi c/\omega$ is the wavelength in vacuum.

I assume $\mu = \mu' = \mu_0$, i.e., nonmagnetic media. The incident medium is vacuum, so $n = 1$. On the transmission side, the plasma medium is described by a dielectric function ϵ' (with $\omega = \omega_p/\sqrt{2}$) and index of refraction n',

$$\epsilon' = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega_p^2/2}\right) = -\epsilon_0, \quad n' = \frac{c}{v'} = \frac{\sqrt{\epsilon'\mu'}}{\sqrt{\epsilon_0\mu_0}} = i$$

For normal incidence, the Fresnel formulas do not depend on the wave polarization, and give for the transmitted amplitude (just inside the plasma):

$$\frac{E_0'}{E_0} = \frac{2n}{n + \frac{\mu}{\mu'}n'} = \frac{2n}{n + n'} = \frac{2}{1+i}$$

The waves propagate in the plasma as $\exp(ik'z)$, where $k' = \omega\sqrt{\epsilon'\mu'} = n'\omega/c = i\omega/c$. So the space dependence of the waves in the plasma is

$$E'(z) = E_0' e^{ik'z} = E_0' e^{i(i\frac{\omega}{c})z} = E_0' e^{-\frac{\omega}{c}z}$$

But $\omega/c = 2\pi f/c = 2\pi/\lambda$, where λ is the vacuum wavelength, and inserting the result from the Fresnel formula,

$$E'(z) = \frac{2E_0}{1+i} e^{-\frac{2\pi}{\lambda}z}$$

For a distance of $z = 5\lambda$ one then has

$$E'(5\lambda) = \frac{2E_0}{1+i} e^{-10\pi} = E_0\,(1-i)\,e^{-10\pi}$$

The magnitude is also found to be

$$|E'(5\lambda)| = E_0\,\sqrt{2}\,e^{-10\pi} \approx (3.2 \times 10^{-14})E_0.$$

Essentially, by this distance into the plasma there is no significant electric field.