GRP-229/CALT-68-1666
August 1990

Inflation and "Reheating" in a Scalar Field Dominated Cosmological Model

Bharat Ratra

Theoretical Astrophysics and Theoretical Physics,
California Institute of Technology, Pasadena, CA 91125

ABSTRACT

A spatially flat cosmological model dominated by a scalar field (Φ) with a tripartite scalar field potential: $V(\Phi) \propto \exp(-\Phi/\sqrt{p})$, where $p > 1$ (for $\Phi < \Phi^{(1)}$ — inflation); $V(\Phi) = m^2(\Phi - \Phi^{(2)})^2$ where m is the scalar field mass (for $\Phi^{(1)} < \Phi < \Phi^{(2)}$ — "reheating"); and $V(\Phi) = 0$ (for $\Phi > \Phi^{(2)}$ — kinetic energy dominance), is a simple, albeit partially unrealistic, phenomenological example of a universe which inflates and "reheats". The transitions in this model (e.g., from inflation to "reheating") are simpler than those in models we have previously studied — in particular, the parameters of the potential are chosen so as to ensure that it does not jump at the transitions and there is no conversion of matter at the transitions (they are between two different scalar field dominated epochs rather than, for instance, between a scalar field dominated inflation epoch and a radiation dominated epoch). Using techniques we have previously developed, we derive solutions of the synchronous gauge relativistic linear perturbation equations of motion during kinetic energy dominance (solutions for the inflation and "reheating" epochs have been presented elsewhere). Joining conditions for these solutions are derived by requiring that the equations of motion be singularity-free on the transition spatial hypersurfaces (which are taken to be spatial hypersurfaces on which the scalar field is spatially homogeneous). The joining conditions are used to determine the constants of integration in the posttransition solutions (the constants of integration in the inflation epoch solutions are determined from quantum mechanically motivated initial conditions). These results are used to derive the late time large-scale energy density power spectrum and gravitational wave two-point function. We find that the spatial momentum dependence of these statistics are determined by the model of the inflation epoch and do not depend on the manner in which the late time universe is modelled. As the inflation model approaches the de Sitter inflation limit the late time large-scale power spectrum approaches the scale-invariant form and the numerical prefactor "diverges". The "divergence" is consistent with what is expected from our analysis of other examples of the inflation scenario and reinforces our earlier conclusion that it is premature to claim that this scenario of the very early universe has a "fine-tuning" problem.

Submitted to *Physical Review D*

1. Introduction and Summary

We have recently examined the behaviour of spatial irregularities in simplified inflation modified hot big bang models, [1-3]. The evolutionary history of these models was divided into three eras: the early time scalar field dominated inflation epoch and the late time radiation and baryon dominated epochs. In each epoch we ignored spatial irregularities in all but the dominant form of matter. Solutions of the synchronous gauge relativistic linear perturbation equations, which govern the evolution of spatial irregularities, could then be derived for each epoch. The constants of integration in the inflation epoch solutions were determined from the quantum mechanical initial conditions of Refs. [4,5], while those in the radiation and matter dominated solutions were determined from joining conditions at the transitions, [6]. We require joining conditions in these models since we have only accounted for perturbations in the dominant form of matter in each epoch. In a more realistic approximation one would keep track of perturbations in all forms of matter present; however, at present, such a model is analytically intractable. The more realistic model could be studied numerically. However, most simple inflation models have at least three free parameters (the models discussed in Refs. [7,8,1] have only two), [3], so to determine the complete free-parameter dependence through numerical analysis is a fairly involved task.

In Refs. [1,3] we made use of joining conditions at both the "reheating" (inflation-radiation) and radiation-baryon transitions. Our analytical treatment of the radiation-baryon transition can be improved upon because the physics of this transition is fairly well understood. Interestingly, the "reheating" transition has not yet been studied in any detail; at best, one has some understanding of this transition in a few simplified models (even in these examples, since the relevant physics is summarized by nonequilibrium quantum statistical field theory in a nontrivial background field, there has not yet been much quantitative study). At present, there is no theory of "reheating", [9], all one has are the models. In a first approximation one might attempt to encode the large-scale features of the "reheating" transition model in a set of joining conditions. In the future one would like to know whether these joining conditions follow from microphysical considerations. For the present, varying the joining condition prescription should allow for a determination of those late time large-scale characteristics of an inflation model that are independent of the microphysical details of "reheating", [9]. The joining conditions for the perturbations are derived by requiring that the equations of motion be singularity-free on the transition hypersurface, [6]. In Refs. [1,3] the transition hypersurface was taken to be the spatial hypersurface on which the total energy density was spatially homogeneous (and which corresponds to the synchronous gauge constant time spatial hypersurface on which the spatially homogeneous part of the energy density assumed the value appropriate for a transition). The joining conditions derived using this prescription differ from other sets of joining conditions found in the literature, [6].

In Ref. [1] we found that observational data does not unduly constrain the parameters of the exponential potential scalar field inflation model of Refs. [7,8,1] (recent discussions of related inflation models include Refs. [10]), while in Ref. [3] we argued for a similar result for the quartic potential scalar field inflation model. In particular, we found that rough consistency with the observations does not require that a parameter of the inflation model scalar field potential assume an exceedingly small value ($\lesssim 10^{-11}$) as had previously been found necessary in a related model, [11]. The reason for this difference is the difference in the prescriptions used to implement "reheating" (in Refs. [11] and in Refs. [1,3]), [9,12].

There are two features of the transition prescription, used in most analyses of the evolution of inflation generated spatial irregularities (including Refs. [1,3]), that deserve to be underscored — first of all, a transition was modelled as an instantaneous, discontinuous, change in the equation of state at the appropriate spatial hypersurface and secondly, at the transition matter was assumed to convert (for instance, from a scalar field to radiation at the inflation-radiation transition). Both of these approximations are, of course, physically unrealistic but, one might hope, not too inappropriate for a preliminary analysis.

The main purpose of this paper is to examine whether a change in the transition prescription affects the gross properties of the final large-scale energy density perturbation power spectrum. We consider a scalar field (Φ) dominated cosmological model divided into three epochs, each characterized by a different functional form for the scalar field potential $V(\Phi)$. The inflation epoch ($\Phi < \Phi^{(1)}$) is described by the exponential potential $V \propto \exp(-\Phi/\sqrt{p})$ (with $p > 1$), [7,8,1]; "reheating" ($\Phi^{(1)} < \Phi < \Phi^{(2)}$) is characterized by the massive scalar field potential $V = m^2 \left(\Phi - \Phi^{(2)}\right)^2$ (where m is the scalar field mass), [13]; and the kinetic energy dominated epoch ($\Phi > \Phi^{(2)}$) has $V = 0$. The parameters of the potential are chosen so that the pieces of the potential "line up" at a transition — this reduces the number of free parameters which characterize the model (derivatives of the parts of the potential are not forced to match at a transition). This model is a simple "reheating" inflation cosmology (the latter part of which is not realized in the real universe, and has been chosen to simplify the computation). Instead of using the spatially homogeneous energy density transition prescription of Refs. [1,3] we choose to define the transition hypersurface as a hypersurface on which the scalar field is spatially homogeneous.

Two features of the model are of special interest — first of all, unlike the models of Ref. [1,3] (and almost all earlier models) there is no conversion between different forms of matter at a transition, and secondly, since we choose to line up the pieces of the scalar field potential at a transition, the joining conditions ensure that the spatially homogeneous parts of the scalar field energy density and pressure do not jump at a transition (as a result the transitions in this model are gentler than those in the models studied in Refs. [1,3]).

We make use of the synchronous gauge formalism of Ref. [14] to analyze this model since it is more amenable to physical interpretation than is the gauge-invariant formalism (recent discussions of variants of the gauge-invariant approach include Refs. [15]).

Prior to summarizing our analysis we highlight some of our results. We shall show that the spatial momentum dependence of the large-scale late time energy density power spectrum and gravitational wave two-point correlation function is determined by the inflation epoch model and is independent of the model used to describe the late time universe. In the limit in which the inflation model approaches the de Sitter (or exponential expansion) model we find that the energy density power spectrum approaches the scale-invariant form, [16], characteristic of de Sitter inflation, [1,3]. In this limit the numerical prefactor of the large-scale late time power spectrum "diverges" in a manner consistent with the results of Refs. [1,3] — this "divergence" also seems to depend only on the model of the inflation epoch. We emphasize, however, that the value of the numerical prefactor does also depend on the transition model as well as on the late-time cosmological model.

The massive scalar field "reheating" model has been studied in Ref. [13]. This model does not account for the small-scale microphysics of the "reheating" transition (in particular, we have ignored effects arising from the coupling of the scalar field to other fields) and is only meant to allow for an examination of the large-scale effect of this transition on the evolution of spatial inhomogeneities generated during a very early epoch of inflation. In Sec. 5.1 we show that the assumptions made in the approximate analysis (JWKB in the scalar field mass) of Ref. [13] are satisfied in the cosmological model considered here. In particular, we show that, during "reheating", it is not unrealistic for the (fast) time scale on which the spatially homogeneous part of the scalar field varies, m^{-1}, to be very much shorter than the relevant cosmological time scale, H^{-1} (where H is the Hubble parameter), provided that the redshift of "reheating" is not too large and provided that "reheating" occurs sufficiently rapidly. In the models of Refs. [1,3] "reheating", i.e., the inflation-radiation transition, is assumed to occur instantaneously when the local energy density drops to an appropriate value. The model considered here, in which the "reheating" transition timescale is finite but still much shorter than the cosmological timescale at this epoch, is more realistic. The de Sitter inflation limit results found here agree with those found in Refs. [1,3]; in particular, the energy density power spectrum increases as the slope of the inflation epoch scalar field potential is decreased, and as a result, late time, large-scale, observational data cannot require that the slope of the inflation epoch scalar field potential be exceedingly small.

Making use of the joining conditions for the spatially homogeneous parts of the fields, we show (in Sec. 5.1) that, for large-scale purposes, the cosmological model is characterized by three "free" parameters — z_1 (the redshift of the inflation-"reheating" transition), z_2 (the redshift of the "reheating" to kinetic energy dominance transition) and p (the parameter in the inflation epoch scalar field potential). Here we have assumed that the current value of the Hubble parameter, $H_{\rm now}$, is known and that the scalar field is the only matter constituent during kinetic energy dominance, i.e., that the cosmological model is spatially flat. To satisfy the assumption of Ref. [13] z_1 and z_2 must obey $z_1 \times z_2 \ll (m/H_{\rm now})^{2/3}$ or if we choose $H_{\rm now} = 100\, h \,{\rm km\, s^{-1}\, Mpc^{-1}}$,

$$z_1 \times z_2 \ll 10^{38} \left[\frac{m(10^{15}{\rm GeV})}{h} \right]^{2/3} , \tag{1.1}$$

which does not seem too unreasonable.

The late time large-scale energy density perturbation power spectra in the models of Refs. [1,3] "diverge" in the de Sitter inflation limit. In this limit the inflation model reduces to a time-independent cosmological constant model and the background energy density evolves exceedingly slowly (because the background scalar field does the same) — this is the reason the late-time power spectra "diverge", [1,3]. Here, we find, during "reheating", in the de Sitter limit, that the large-scale gauge-invariant part of the power spectrum (the spatial momentum space fractional energy density irregularity two-point correlation function) is

$$P_{\Delta(R)}(k,t) = \frac{1}{\epsilon^2} \frac{16\pi}{m_p^2}\, k\, |\hat{A}_R|^2 (a_R)^2, \tag{1.2}$$

where m_p is the Planck mass, k the coordinate spatial momentum, a_R is the scale factor during "reheating", ϵ $(= 1/\sqrt{p})$ characterizes the slope of the potential during inflation ($\epsilon \to 0$ is the de Sitter limit) and $|\hat{A}_R|$ is an ϵ-independent factor defined in eqn. (6.34) below. During kinetic energy dominance we find, in the de Sitter inflation limit,

$$P_{\Delta(K)}(k,t) = \frac{1}{\epsilon^2}\frac{16\pi}{m_p^2}\,k\,|\hat{A}_K|^2(a_K)^8, \qquad (1.3)$$

where a_K is the scale factor during kinetic energy dominance and the ϵ-independent factor $|\hat{A}_K|$ is defined in eqn. (6.38). The reason these power spectra increase as ϵ is decreased is related to the reason the late time power spectra "diverge" (in the de Sitter inflation limit) in the models of Refs. [1,3] — a "slowly rolling", inflation epoch, background scalar field (to make this connection here is not as straightforward as it was in Refs. [1,3] — the reasons for this are discussed in Sec. 6.3). The ϵ dependence of eqns. (1.2) and (1.3) mean that an upper bound on the power spectrum can only result in a lower bound on ϵ.

(More precisely, rapid reheating inflation models do not require a small coupling constant to be consistent with observational constraints because the power spectrum is a sensitive function of both the redshift of reheating and the coupling constant. For instance, in the scale-invariant limit of the model considered in Ref. [1], at the present epoch, and in a fairly accurate approximation,

$$P_\Delta \propto \frac{(1+z_{\rm re})^4}{\epsilon^2}, \qquad (1.4)$$

where $z_{\rm re}$ is the redshift of reheating. Clearly, one may reduce the magnitude of P_Δ by decreasing $(1+z_{\rm re})$. In the slow reheating model considered in Refs. [11], the form of the potential at reheating ensures that P_Δ only depends on the logarithm of the redshift of reheating, and for non-extreme values of this parameter the coupling constant is significantly constrained by observational data. This is discussed further in Refs. [9,12]; also see the model studied in Sec. III,B of the last of Refs. [11], where the power spectrum depends on the square, and not the logarithm, of the intermediate mass scale.)

A brief description of the spatially flat scalar field dominated cosmological model is given in Sec. 2. Sec. 3 is an examination of the third, scalar field kinetic energy dominated, epoch of the model. We study the evolution of the spatially homogeneous part of the metric and scalar field in Sec. 3.1 and in Sec. 3.2 we examine the behaviour of spatial irregularities. In Sec. 3.3 we derive expressions for perturbations in the equivalent spacetime-dependent "speed of sound" fluid model, and compare these expressions to those in the ideal fluid model with equation of state $p = \rho$. The remnants of general coordinate invariance (in synchronous gauge) is used, in Sec. 3.4, to catalogue the gauge-invariant and gauge-dependent terms in the solutions of Secs. 3.2 and 3.3 (gauge-invariant combinations of the spatial irregularities are also tabulated in this section). Sec. 3.5 concludes our examination of this epoch with a discussion of the large-scale and small-scale expansions of the expressions for spatial irregularities. Sec. 4 deals with the derivation of the joining conditions that follow on requiring that the equations of motion be singularity-free on the transition hypersurface. Although the transition hypersurface used here differs from that used in Ref. [6], the general procedures are similar. Since it seems our discussion in Ref. [6] was a little too sketchy, we elaborate on aspects of the general procedure in this section — in particular, joining conditions for the spatially homogeneous fields are derived in Sec. 4.1, Sec. 4.2 deals with the joining conditions for spatial irregularities while Sec. 4.3 is a tabulation of the joining conditions for perturbations in the equivalent fluid model (as well as those for gauge-invariant variables). In Sec. 5 we use these joining conditions to determine the constants of integration in the large time solutions; in Sec. 5.1 we fix the constants in the spatially homogeneous parts of the fields (and list the free parameters of the large-scale cosmological model) while in Secs. 5.2 and 5.3 we determine those in the "reheating" and kinetic energy dominated epoch spatial irregularity solutions. In Sec. 6 we focus on the large-scale spatial momentum space two-point correlation functions for energy density and gravitational wave perturbations during "reheating" and kinetic energy dominance. We first, in Sec. 6.1, determine the form the constants assume when restricted to scales larger than the Hubble scale during the transitions. These results are used in Sec. 6.2 to derive the late time large-scale two-point correlation functions which are then examined in the de Sitter inflation limit in Sec. 6.3.

2. The Model

The Einstein-scalar field action is

$$S = \frac{m_p^2}{16\pi} \int dt\, d^3x \sqrt{-g} \left[-R + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} V(\Phi) \right] . \tag{2.1}$$

The metric signature is taken to be $(+ - - -)$. The scalar field potential is

$$V(\Phi) = V_I(\Phi) \left[1 - \Theta\Big(\Phi - \Phi^{(1)}\Big) \right] + V_R(\Phi) \left[\Theta\Big(\Phi - \Phi^{(1)}\Big) - \Theta\Big(\Phi - \Phi^{(2)}\Big) \right] , \tag{2.2}$$

where the Heaviside step function is defined by

$$\Theta(\Phi) = \begin{cases} 0 & \Phi < 0 \\ 1 & \Phi > 0. \end{cases} \tag{2.3}$$

The scalar field potential during inflation (when $\Phi < \Phi^{(1)}$), [8,1], is

$$V_I(\Phi) = \left(\frac{6-q}{3} \right) \frac{16\pi}{m_p^2} \rho_{b\Phi}^{(0)} \exp \left[-\sqrt{\frac{q}{2}} \Big(\Phi - \Phi^{(0)} \Big) \right] \tag{2.4}$$

[here $\Phi^{(0)}$ $(< \Phi^{(1)})$ is the scalar field when the scale factor $a = a_0$, $\rho_{b\Phi}^{(0)}$ is the scalar field background energy density at the same time and $q\,(= 2/p)$ is a parameter, the relevant range being $0 < q < 2$], while the potential during "reheating" (when $\Phi^{(1)} < \Phi < \Phi^{(2)}$), [13], is

$$V_R(\Phi) = m^2 \Big(\Phi - \Phi^{(2)} \Big)^2 \tag{2.5}$$

where m is the scalar field mass. The potential during kinetic energy dominance (when $\Phi > \Phi^{(2)}$) vanishes. Continuity requirements at the transition values of the field ($\Phi^{(1)}$ and $\Phi^{(2)}$) relate some of the parameters in these expressions (this is discussed in Secs. 4 and 5).

3. Kinetic Energy Dominance

The equations governing the evolution of the model are most conveniently recorded by retaining the lowest and first order terms of the Einstein equations and the scalar field equation of motion linearized about a spatially flat FLRW model and a spatially homogeneous scalar field, [8]. We work in synchronous gauge so the line element is taken to be

$$ds^2 = dt^2 - a^2(t)\,[\delta_{ij} - h_{ij}(t,\vec{x})]\,dx^i dx^j\,, \tag{3.1}$$

where the metric perturbations are denoted by h_{ij}; the expansion for the scalar field is

$$\Phi(t,\vec{x}) = \Phi_b(t) + \phi(t,\vec{x}) \tag{3.2}$$

where Φ_b is the spatially homogeneous part and ϕ is the scalar field spatial irregularity.

In the following sections we study the scalar field model with vanishing potential (this is a special case, $q = 6$, of the model studied in Ref. [1]; since some of the solutions simplify we have reanalyzed this case). Related models have been examined in Refs. [17].

3.1 Spatially Homogeneous Background Fields

The equations of motion for the spatially homogeneous part of the fields are

$$\ddot{\Phi}_b + 3\,\frac{\dot{a}}{a}\,\dot{\Phi}_b + \frac{1}{2}V'(\Phi_b) = 0, \tag{3.3}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{12}\left[(\dot{\Phi}_b)^2 + V(\Phi_b)\right], \tag{3.4}$$

$$\frac{\ddot{a}}{a} = -\,\frac{1}{12}\left[2(\dot{\Phi}_b)^2 - V(\Phi_b)\right], \tag{3.5}$$

where a dot denotes a derivative with respect to time.

During kinetic energy dominance (when $\Phi > \Phi^{(2)}$) the scalar field potential and its derivatives vanish, eqn. (2.2), and the first integral of eqn. (3.3) is

$$\dot{\Phi}_{bK}(t) = \frac{C^{(K)}}{a^3}, \tag{3.6}$$

where the sub or superscript K will mean that the corresponding variable is the one in the kinetic energy dominated epoch and $C^{(K)}$ is a constant of integration which will be determined from the appropriate initial/joining conditions in Sec. 5. Using eqn. (3.6) we find that the (expanding) solution of eqn. (3.4) is

$$a(t) = a_2 \left[1 + \frac{\sqrt{3}}{2} \frac{C^{(K)}}{(a_2)^3} (t - t_2) \right]^{1/3}, \tag{3.7}$$

where a_2 is the value of the scale factor at the time $t = t_2$ (the time of the "reheating" to kinetic energy dominance transition). It may easily be seen that the solutions (3.6) and (3.7) satisfy eqn. (3.5). From eqns. (3.6) and (3.7) we have

$$\Phi_{bK}(t) = \Phi^{(2)} + \frac{2}{\sqrt{3}} \ln \left[1 + \frac{\sqrt{3}}{2} \frac{C^{(K)}}{(a_2)^3} (t - t_2) \right], \tag{3.8}$$

where $\Phi^{(2)}$ is the value of the scalar field at the time $t = t_2$.

3.2 Spatial Irregularities

The first order perturbation equations (Sec. 7 of Ref. [8]) are

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + \frac{1}{2}V''(\Phi_b)\phi = \frac{1}{2}\dot{h}\dot{\Phi}_b, \tag{3.9}$$

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} = 2\dot{\Phi}_b\dot{\phi} - \frac{1}{2}V'(\Phi_b)\phi, \tag{3.10}$$

$$\dot{h}_{,i} - \dot{h}_{ij,j} = \dot{\Phi}_b \partial_i \phi, \tag{3.11}$$

$$\frac{1}{a^2}(h_{ij,kk} + h_{,ij} - h_{ik,jk} - h_{jk,ik}) - 3\frac{\dot{a}}{a}\dot{h}_{ij} - \frac{\dot{a}}{a}\dot{h}\,\delta_{ij} - \ddot{h}_{ij} = \frac{1}{2}\delta_{ij}V'(\Phi_b)\phi, \tag{3.12}$$

where the trace of the metric perturbation is denoted by h, spatial indices are raised and lowered with δ_{ij}, and $\nabla^2 \equiv \delta_{ij}\partial_i\partial_j$. We shall work in spatial coordinate momentum space,

$$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[\phi(t, \vec{k})e^{i\vec{k}\cdot\vec{x}} + \phi^*(t, \vec{k})e^{-i\vec{k}\cdot\vec{x}} \right], \tag{3.13}$$

$$h_{ij}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[h_{ij}(t, \vec{k})e^{i\vec{k}\cdot\vec{x}} + h^*_{ij}(t, \vec{k})e^{-i\vec{k}\cdot\vec{x}} \right], \tag{3.14}$$

where $\vec{k}$ is the spatial coordinate momentum ($-\infty \le k_i \le \infty$) and $\vec{k} \cdot \vec{x} = \delta_{ij}k_i x_j$. In what follows we make use of the notation $k \equiv |k_i k_i|^{1/2}$. It also proves to be convenient to change from the independent variable t (time) to x, where x is defined through

$$\frac{\sqrt{3}}{2}\frac{C^{(K)}}{(a_2)^3}e^x = 1 + \frac{\sqrt{3}}{2}\frac{C^{(K)}}{(a_2)^3}(t-t_2). \tag{3.15}$$

During kinetic energy dominance, the equations of motion for the spatial irregularities ϕ and h, eqns. (3.9) and (3.10), may be reexpressed (in Fourier space) as

$$\phi''(x) + \bar{\lambda}^2 e^{4x/3}\phi(x) = \frac{1}{\sqrt{3}}h'(x)\,, \tag{3.16}$$

$$h''(x) - \frac{1}{3}h'(x) = \frac{4}{\sqrt{3}}\phi'(x)\,, \tag{3.17}$$

where a prime represents a derivative with respect to x and we have defined

$$\bar{\lambda} = k\left(\frac{2}{\sqrt{3}\,C^{(K)}}\right)^{1/3}\,. \tag{3.18}$$

Eqn. (3.17) may be used to eliminate h' from eqn. (3.16); the resulting equation is

$$\phi'''(x) - \frac{1}{3}\phi''(x) + \left[-\frac{4}{3} + \bar{\lambda}^2 e^{4x/3}\right]\phi'(x) + \bar{\lambda}^2 e^{4x/3}\phi(x) = 0. \tag{3.19}$$

Since we have differentiated eqn. (3.16) to derive this equation we will have to check whether all the solutions of eqn. (3.19) satisfy eqn. (3.16).

Using the techniques of Ref. [14] we find that the general solution of eqn. (3.19) is

$$\phi_K(x) = c_2^{(K)}e^{-x} - \frac{3}{2}\lambda_K^{-3/2}e^{-x/3}\left(c_+^{(K)}G_1^K(x) + c_-^{(K)}G_2^K(x)\right)\,, \tag{3.20}$$

where we have defined the functions

$$G_1^K(x) = H_0^{(1)}\left(\lambda_K\,e^{2x/3}\right)\,S_{3/2,1}\left(\lambda_K\,e^{2x/3}\right) - \frac{3}{2}\,H_1^{(1)}\left(\lambda_K\,e^{2x/3}\right)\,S_{1/2,0}\left(\lambda_K\,e^{2x/3}\right)\,, \tag{3.21}$$

$$G_2^K(x) = H_0^{(2)}\left(\lambda_K\,e^{2x/3}\right)\,S_{3/2,1}\left(\lambda_K\,e^{2x/3}\right) - \frac{3}{2}\,H_1^{(2)}\left(\lambda_K\,e^{2x/3}\right)\,S_{1/2,0}\left(\lambda_K\,e^{2x/3}\right)\,, \tag{3.22}$$

where

$$\lambda_K = \frac{3}{2}\bar{\lambda}\,, \tag{3.23}$$

$H_\mu^{(i)}$ are Hankel functions and $S_{\mu,\nu}$ are Lommel functions. In eqn. (3.20) we have chosen the normal modes so that $G_2^K(x) = (G_1^K(x))^*$ and $c_2^{(K)}, c_\pm^{(K)}$ are k dependent constants of integration which must be determined from the appropriate initial/joining conditions. We shall see that the term proportional to $c_2^{(K)}$ in eqn. (3.20) is the gauge-dependent time translation solution while the terms proportional to $c_{\pm}^{(K)}$ are linear combinations of the dominant and subdominant adiabatic solutions.

Using eqn (3.20) we find that the general solution of eqn. (3.17) is

$$h^{(K)}(x) = c_1^{(K)} - \sqrt{3}\, c_2^{(K)} e^{-x} + \frac{3}{2}\sqrt{3}\, \lambda_K^{-3/2} e^{-x/3} \left(c_+^{(K)} F_1^K(x) + c_-^{(K)} F_2^K(x) \right) + c_5^{(K)} e^{x/3}, \tag{3.24}$$

where $c_1^{(K)}$ and $c_5^{(K)}$ are k dependent constants of integration,

$$\begin{aligned} F_1^K(x) =& H_0^{(1)}\left(\lambda_K\, e^{2x/3}\right) \left\{ \left(1 - \frac{4}{3}\lambda_K^2\, e^{4x/3}\right) S_{3/2,1}\left(\lambda_K\, e^{2x/3}\right) + \frac{4}{3}\lambda_K^{5/2} e^{5x/3} \right\} \\ & - \frac{3}{2}\, H_1^{(1)}\left(\lambda_K\, e^{2x/3}\right) \left\{ \left(1 - \frac{4}{3}\lambda_K^2\, e^{4x/3}\right) S_{1/2,0}\left(\lambda_K\, e^{2x/3}\right) + \frac{4}{3}\lambda_K^{3/2} e^{x} \right\} \end{aligned} \tag{3.25}$$

and $F_2^{(K)}(x) = (F_1^{(K)}(x))^*$. We shall see that the $c_1^{(K)}$ solution in eqn. (3.24) corresponds to the gauge arbitrariness in rescaling a_2 and that $c_5^{(K)}$ must be related to $c_2^{(K)}$.

Considering a plane wave with propagation vector $\vec{k}$ and choosing the x^1 axis to lie along the spatial momentum direction (and denoting the transverse directions by $x^I, I = 2,3$) we find that eqns. (3.11) and (3.12) reduce to

$$h'_{I1}(x) = 0, \tag{3.26}$$

$$h''_{I1}(x) = 0, \tag{3.27}$$

$$h''_+(x) + \bar{\lambda}^2 e^{4x/3} h_+(x) = 0, \tag{3.28}$$

$$h''_\times(x) + \bar{\lambda}^2 e^{4x/3} h_\times(x) = 0, \tag{3.29}$$

$$h'_{22}(x) + h'_{33}(x) = \frac{2}{\sqrt{3}}\phi(x), \tag{3.30}$$

$$h''_{22}(x) + h''_{33}(x) = \frac{2}{\sqrt{3}}\phi'(x), \tag{3.31}$$

$$\bar{\lambda}^2 e^{4x/3}(h_{22}(x) + h_{33}(x)) = -\frac{2}{3}(\sqrt{3}\phi'(x) + h'(x)), \tag{3.32}$$

where $h_+ = h_{22} - h_{33}$ and $h_\times = h_{23} + h_{32}$ are the transverse gravitational wave perturbations.

The solutions of eqns. (3.26) and (3.27) are

$$h_{I1}^{(K)}(x) = c_{I,1}^{(K)}, \tag{3.33}$$

where $c_{I,1}^{(K)}$ are k dependent constants of integration. These solutions are gauge-dependent. The solutions of the gravitational wave equations of motion, eqns. (3.28) and (3.29), are

$$h_i^{(K)}(x) = c_{i1}^{(K)} H_0^{(1)}\left(\lambda_K\, e^{2x/3}\right) + c_{i2}^{(K)} H_0^{(2)}\left(\lambda_K\, e^{2x/3}\right), \tag{3.34}$$

where i is either $+$ or $\times$ and $c_{i1}^{(K)}$ and $c_{i2}^{(K)}$ are k dependent constants of integration. These solutions are gauge-invariant. The general solution of eqns. (3.30) and (3.31) is

$$h_{22}^{(K)}(x)+h_{33}^{(K)}(x) = -\frac{2}{\sqrt{3}}c_2^{(K)}e^{-x}+\sqrt{3}\lambda_K^{-3/2}e^{-x/3}\left(c_+^{(K)}\widetilde{F}_1^K(x) + c_-^{(K)}\widetilde{F}_2^K(x)\right)+c_7^{(K)}, \tag{3.35}$$

where $c_7^{(K)}$ is a k dependent constant of integration (we shall see that it must vanish),

$$\widetilde{F}_1^K(x) = G_1^K(x) - \lambda_K^{1/2}e^{x/3}H_0^{(1)}\left(\lambda_K\, e^{2x/3}\right), \tag{3.36}$$

$\widetilde{F}_2^K(x) = (\widetilde{F}_1^K(x))^*$ and $G_1^K(x)$ is defined in eqn. (3.21). The $c_2^{(K)}$ term in eqn. (3.35) is the gauge-dependent time translation solution while the $c_\pm^{(K)}$ terms correspond to linear combinations of dominant and subdominant adiabatic perturbations.

Using eqn. (3.35) we find that the left hand side of eqn. (3.32) is given by

$$\begin{aligned}\bar{\lambda}^2\, e^{4x/3}(h_{22}^{(K)}(x) + h_{33}^{(K)}(x)) = &-\frac{8}{9\sqrt{3}}\,\lambda_K^2\, c_2^{(K)}e^{x/3} + \frac{4}{9}\,\lambda_K^2\, c_7^{(K)}e^{4x/3} \\ &- \sqrt{3}\lambda_K^{-3/2}e^{-x/3}\left(c_+^{(K)}\hat{F}_1^K(x) + c_-^{(K)}\hat{F}_2^K(x)\right),\end{aligned} \tag{3.37}$$

where we have defined

$$\hat{F}_1^K(x) = -\frac{4}{9}\lambda_K^2\, e^{4x/3}\widetilde{F}_1^K(x), \tag{3.38}$$

$\hat{F}_2^K(x) = (\hat{F}_1^K(x))^*$. Using eqns. (3.20) and (3.24) the right hand side of eqn. (3.32) is

$$\begin{aligned}-\frac{2}{3}\left[\left(h^{(K)}\right)'(x) + \sqrt{3}\phi_K'(x)\right] = &-\frac{2}{9}c_5^{(K)}\, e^{x/3} \\ &- \sqrt{3}\lambda_K^{-3/2}e^{-x/3}\left(c_+^{(K)}\hat{F}_1^K(x) + c_-^{(K)}\hat{F}_2^K(x)\right).\end{aligned} \tag{3.39}$$

Comparing eqns. (3.37) and (3.39) we see that $c_7^{(K)}$ must vanish and that $c_5^{(K)}$ must be related to $c_2^{(K)}$ through

$$c_5^{(K)} = \frac{4}{\sqrt{3}}\,\lambda_K^2\,c_2^{(K)}. \tag{3.40}$$

Using this, it may be shown that eqns. (3.20) and (3.24) satisfy eqns. (3.16) and (3.17). The perturbed spatial scalar curvature is given by

$$\begin{aligned} {}^{(3)}R &= \frac{k^2}{a^2}(h_{22}(x) + h_{33}(x)) \qquad (3.41)\\ &= -\frac{8}{9\sqrt{3}}\,\lambda_K^2\,c_2^{(K)} e^{-5x/3} + \frac{4}{3\sqrt{3}}\,\lambda_K^{1/2}\,e^{-x}\left(c_+^{(K)}\widetilde{F}_1^K(x) + c_-^{(K)}\widetilde{F}_2^K(x)\right), \end{aligned}$$

where we have set $c_7^{(K)} = 0$ and $\widetilde{F}_1^K(x)$ and $\widetilde{F}_2^K(x)$ are defined by eqn. (3.36).

In summary, the solutions of the perturbation equations of motion, eqns. (3.9) – (3.12), are given in eqns. (3.20), (3.24) and (3.33) – (3.35) where $c_7^{(K)}$ vanishes and $c_5^{(K)}$ is related to $c_2^{(K)}$ by eqn. (3.40). The ten k dependent constants of integration in these solutions must be determined from the appropriate initial/joining conditions.

3.3 Equivalent Spacetime-Dependent “Speed of Sound” Fluid Model

The energy density and pressure of the spatially homogeneous part of the scalar field are

$$\rho_{b\Phi} = \frac{m_p^2}{32\pi}\left[(\dot{\Phi}_b)^2 + V(\Phi_b)\right], \tag{3.42}$$

$$p_{b\Phi} = \frac{m_p^2}{32\pi}\left[(\dot{\Phi}_b)^2 - V(\Phi_b)\right]. \tag{3.43}$$

From eqns. (3.6) and (2.2) we find, during kinetic energy dominance,

$$\rho_{b\Phi(K)} = \frac{m_p^2}{32\pi}\left(C^{(K)}\right)^2\frac{1}{a^6} = p_{b\Phi(K)}; \tag{3.44}$$

a spatially homogeneous massless scalar field has a rather stiff equation of state.

Scalar field perturbations behave like perturbations in a fluid with a spacetime-dependent “speed of sound”, [6]; linear combinations of the scalar field irregularity and its spatial and temporal derivatives provide operational definitions of the corresponding perturbed fluid variables, [8]. The perturbation in the energy density is (eqn. (7.10) of Ref. [8])

$$\rho_{b\Phi}\delta_\Phi = \frac{m_p^2}{16\pi}\left[\dot{\Phi}_b\dot{\phi} + \frac{1}{2}V'(\Phi_b)\phi\right] \tag{3.45}$$

(here δ_Φ is the fractional energy density perturbation), and the proper peculiar velocity perturbation is (eqn. (7.11) of Ref. [8])

$$(\rho_{b\Phi} + p_{b\Phi})\, v^i_\Phi = -\frac{m_p^2}{16\pi}\frac{\dot{\Phi}_b}{a}\,\partial_i\phi. \tag{3.46}$$

The effective "speed of sound" $c_{s\Phi}$ is defined by eqns. (3.45) above and (7.12) of Ref. [8],

$$c^2_{s\Phi}\,\rho_{b\Phi}\,\delta_\Phi = \frac{m_p^2}{16\pi}\left[\dot{\Phi}_b\dot{\phi} - \frac{1}{2}V'(\Phi_b)\phi\right]. \tag{3.47}$$

In the massless scalar field model we have

$$c_{s\Phi(K)} = 1. \tag{3.48}$$

It is coincidental that the value of this speed is the same as the value of the speed that appears in the scalar field perturbation dispersion relation. From eqns. (3.6), (3.20), (3.44) and (3.45) we find, in the kinetic energy dominated model,

$$\delta_{\Phi(K)}(x) = -\sqrt{3}\,c_2^{(K)}e^{-x} + \frac{3}{2}\sqrt{3}\,\lambda_K^{-3/2}e^{-x/3}\left(c_+^{(K)}\hat{G}_1^K(x) + c_-^{(K)}\hat{G}_2^K(x)\right), \tag{3.49}$$

where

$$\hat{G}_1^K(x) = G_1^K(x) + \frac{2}{3}\lambda_K^{3/2}e^x H_1^{(1)}\left(\lambda_K\, e^{2x/3}\right), \tag{3.50}$$

$\hat{G}_2^K(x) = (\hat{G}_1^K(x))^*$ and $G_1^K(x)$ is defined in eqn. (3.21). Similarly we find that the longitudinal peculiar velocity perturbation is given by

$$v^1_{\Phi(K)}(x) = -\frac{i}{\sqrt{3}}\lambda_K\,c_2^{(K)}e^{-x/3} + i\,\frac{\sqrt{3}}{2}\lambda_K^{-1/2}e^{x/3}\left(c_+^{(K)}G_1^K(x) + c_-^{(K)}G_2^K(x)\right), \tag{3.51}$$

while the transverse peculiar velocity perturbation vanishes,

$$v^I_{\Phi(K)}(x) = 0. \tag{3.52}$$

It is instructive to compare the expressions for perturbations in the massless scalar field model derived here to the expressions for perturbations in the $\nu = 1$ ideal fluid model (where the equation of state is $p_b = \nu \rho_b$) given in Ref. [14]. (Some aspects of the $\nu = 1$ fluid model have been touched upon in Ref. [18].) Comparing the $\nu = 1$ version of eqns. (11) and (12) of Ref. [14] with eqns. (3.44) and (3.7) above we have the identification

$$\rho_b^{(0)} = \frac{m_p^2}{32\pi} \frac{\left(C^{(K)}\right)^2}{(a_2)^6}, \tag{3.53}$$

$$M = \frac{\sqrt{3}}{2} \frac{C^{(K)}}{(a_2)^3}. \tag{3.54}$$

(The sub/superscript 0 in Ref. [14] corresponds to the sub/superscript 2 used in this section.) In the $\nu = 1$ model, with the above identifications, the parameters λ and $\tilde{\nu}$ used in Ref. [14] assume the values

$$\lambda = \lambda_K \quad \text{and} \quad \tilde{\nu} = 0. \tag{3.55}$$

Comparing the expressions for the fractional energy density perturbations (eqn. (3.49) above and the $\nu = 1$ version of eqn. (16) of ref. [14]) we find that they coincide provided the constants of integration used in Ref. [14] are related to those used here through

$$c_2 = -\sqrt{3}\, c_2^{(K)}, \tag{3.56}$$

$$c_\pm = \frac{2}{\sqrt{3}} \lambda_K \, c_\pm^{(K)}. \tag{3.57}$$

In the $\nu = 1$ ideal fluid model $c_s = 1$ which agrees with eqn. (3.48). The expressions for the trace of the metric perturbation (eqn. (3.24) above and the $\nu = 1$ version of eqn. (17) of Ref. [14]) coincide if we use eqns. (3.56) and (3.57) as well as

$$c_5 = c_5^{(K)}, \tag{3.58}$$

$$c_1 = c_1^{(K)}. \tag{3.59}$$

Eqns. (3.56) and (3.58) are consistent with eqn. (3.40) above and the corresponding equation, the $\nu = 1$ version of eqn. (23), of Ref. [14]. The solutions for the metric perturbation combination $h_{22} + h_{33}$ (eqn. (3.35) above and the $\nu = 1$ version of eqn. (31) of Ref. [14]) are identical if eqns. (3.56) and (3.57) are obeyed and we identify

$$c_7 = c_7^{(K)} \tag{3.60}$$

(c_8 in eqn. (31) of Ref. [14] vanishes — see the comments below the equation below eqn. (31) and below eqn. (33) there). The relations (3.56) and (3.57) ensure that the longitudinal peculiar velocity perturbations coincide (eqn. (3.51) above and the $\nu = 1$ version of eqn. (32) of Ref. [14]). The identification

$$c_{I,1} = c_{I,1}^{(K)} \tag{3.61}$$

makes part of the solutions for the h_{I1} components of the metric perturbations coincide (eqn. (3.33) above and the $\nu = 1$ version of eqn. (37) of Ref. [14]) — the $c_{J,2}\, x$ term in eqn. (37) for the $\nu = 1$ model (see comment below eqn. (37) there) does not have a counterpart in the massless scalar field model. This term results in the $\nu = 1$ ideal fluid model transverse peculiar velocity solution (see comment below eqn. (38) of Ref. [14])

$$v^I(x) = \frac{i}{2}\, \lambda\, c_{I,2}\, e^{2x/3}; \tag{3.62}$$

this solution also does not have a counterpart in the massless scalar field model, eqn. (3.52). The gravitational wave perturbations, the $\nu = 1$ versions of eqns. (39) and (40) of Ref. [14] and eqns. (3.34) above, coincide provided we identify

$$c_{i1} = c_{i1}^{(K)}, \tag{3.63}$$

$$c_{i2} = c_{i2}^{(K)}. \tag{3.64}$$

In summary, spatial irregularities in the massless scalar field model coincide with those in the $\nu = 1$ ideal fluid model, apart from the transverse peculiar velocity perturbation (which cannot be supported in a scalar field model, [1]). We note that the gauge-dependent transverse peculiar velocity perturbation in the $\nu = 1$ ideal fluid model, eqn. (3.62), increases with time, while that in the Einstein-de Sitter model decreases with time.

3.4 Remnants of General Coordinate Invariance

Choosing synchronous gauge does not completely fix general coordinate invariance transformations — there are four remaining time-independent gauge symmetries. Their effect on metric perturbations are

$$\begin{aligned}
\delta h_{11}(x,k) &= f^0(k)\,\frac{8}{3}\,\lambda_K^2\,e^{x/3} - 2ik\,\omega_1(k) - f^0(k)\,\frac{2}{3}\,e^{-x},\\
\delta h_{1I}(x,k) &= -ik\,\omega_I(k),\\
\delta h_{IJ}(x,k) &= -f^0(k)\,\frac{2}{3}\,e^{-x}\,\delta_{IJ},\\
\delta h_+(x,k) &= 0 = \delta h_\times(x,k),\\
\delta h(x,k) &= f^0(k)\,\frac{8}{3}\,\lambda_K^2\,e^{x/3} - 2ik\,\omega_1(k) - f^0(k)\,2\,e^{-x},
\end{aligned} \tag{3.65}$$

where the gauge parameters f^0 and ω_i are time (x) independent. The spatial curvature perturbation transformation is given by

$$\delta\left[{}^{(3)}R(x,k)\right] = -f^0(k)\,\frac{16}{27}\,\lambda_K^2\,e^{-5x/3}. \tag{3.66}$$

From the definition and the transformation properties of the stress-energy tensor we find that the scalar field perturbation transforms as

$$\delta\phi(x,k) = f^0(k)\,\frac{2}{\sqrt{3}}\,e^{-x}, \tag{3.67}$$

while the fractional energy density perturbation and the peculiar velocity perturbation, eqns. (3.45) and (3.46), vary as

$$\delta\left[\delta_\Phi(x,k)\right] = -f^0(k)\,2\,e^{-x}, \tag{3.68}$$

$$\delta v_\Phi^1(x,k) = -i\,f^0(k)\,\frac{2}{3}\,\lambda_K\,e^{-x/3}, \tag{3.69}$$

$$\delta v_\Phi^I(x,k) = 0. \tag{3.70}$$

It is straightforward to establish that the choice

$$f^0(k) = \frac{\sqrt{3}}{2}c_2^{(K)}\,, \tag{3.71}$$

generates the solutions proportional to $c_2^{(K)}$ and $c_5^{(K)}$ in eqns. (3.20), (3.24) [where we use eqn. (3.40) to relate $c_5^{(K)}$ to $c_2^{(K)}$], (3.35), (3.41), (3.49) and (3.51). Choosing

$$\omega_1(k) = \frac{i}{2k}\, c_1^{(K)}\,, \tag{3.72}$$

allows one to derive the $c_1^{(K)}$ solution in eqn. (3.24), while the choice

$$\omega_I(k) = \frac{i}{k} c_{I,1}^{(K)} \tag{3.73}$$

generates the solution for h_{I1}, eqn. (3.33). Gravitational wave perturbations, eqn. (3.65), and transverse peculiar velocity perturbations, eqn. (3.70), are gauge-invariant.

In summary, only six of the ten k dependent constants of integration correspond to gauge-invariant solutions. These are the scalar field perturbation constants $c_\pm^{(K)}$ and the gravitational wave perturbation constants $c_{+1}^{(K)}$, $c_{+2}^{(K)}$, $c_{\times 1}^{(K)}$ and $c_{\times 2}^{(K)}$.

Two convenient gauge-invariant variables are Δ_Φ (the gauge-invariant measure of energy density perturbations) and A_Φ, [1,3],

$$\Delta_\Phi = -\frac{m_p^2}{16\pi}\,\frac{1}{\rho_{b\Phi}}\,\left[{}^{(3)}R + \frac{\dot a}{a}\frac{\partial}{\partial t}\left\{\frac{3}{\nabla^2}h_{ij,ij} - h\right\}\right], \tag{3.74}$$

$$A_\Phi = \delta_\Phi - \frac{3}{4}\left(1 + \frac{p_{b\Phi}}{\rho_{b\Phi}}\right)(h_{22} + h_{33}). \tag{3.75}$$

In the massless scalar field model we find

$$\Delta_{\Phi(K)}(x,k) = \sqrt{3}\, e^{2x/3}\left(c_+^{(K)} H_1^{(1)}\left(\lambda_K\, e^{2x/3}\right) + c_-^{(K)} H_1^{(2)}\left(\lambda_K\, e^{2x/3}\right)\right), \tag{3.76}$$

$$A_{\Phi(K)}(x,k) = \frac{3}{2}\sqrt{3}\,\lambda_K^{-1}\left(c_+^{(K)}\bar F_1^K(x) + c_-^{(K)}\bar F_2^K(x)\right), \tag{3.77}$$

where we have defined

$$\bar F_1^K(x) = H_0^{(1)}\left(\lambda_K\, e^{2x/3}\right) + \frac{2}{3}\lambda_K\, e^{2x/3} H_1^{(1)}\left(\lambda_K\, e^{2x/3}\right) \tag{3.78}$$

and $\bar F_2^K(x) = (\bar F_1^K(x))^*$. We note that the time translation solution (the solution proportional to $c_2^{(K)}$) does not contribute to eqns. (3.76) and (3.77).

3.5 Asymptotics

We note that

$$\lambda_K \, e^{2x/3} = \frac{1}{2}\frac{k}{aH} = k \int^t \frac{dt'}{a(t')} = k\tau, \tag{3.79}$$

where H is the Hubble parameter and τ is conformal time.

On small scales (when $k \gg aH$) the scalar field perturbation, eqn. (3.20), oscillates,

$$\phi_K(k,t) = \frac{1}{a}\left[\widetilde{c}_+ \exp\left(ik\int^t \frac{dt'}{a(t')}\right) + \widetilde{c}_- \exp\left(-ik\int^t \frac{dt'}{a(t')}\right)\right], \tag{3.80}$$

where we have ignored the $c_2^{(K)}$ solution in eqn. (3.20) and defined

$$\widetilde{c}_\pm = -\left(\frac{4}{3\pi}\right)^{1/2}\left(\frac{\sqrt{3}C^{(K)}}{2}\right)^{5/6} k^{-3/2}\, e^{\mp i\pi/4}\, c_\pm^{(K)}. \tag{3.81}$$

The $1/a$ dependence of the amplitude of eqn. (3.80) is that expected for a relativistic scalar field. Small-scale gravitational wave perturbations, eqn. (3.34), also oscillate,

$$h_i^{(K)}(k,t) = \frac{1}{a}\left[\widetilde{c}_{i1} \exp\left(ik\int^t \frac{dt'}{a(t')}\right) + \widetilde{c}_{i2} \exp\left(-ik\int^t \frac{dt'}{a(t')}\right)\right] \tag{3.82}$$

[the $1/a$ dependence of the amplitude is that expected for gravitational waves], where

$$\widetilde{c}_{i\binom{1}{2}} = \left(\frac{2}{\sqrt{3}}\frac{C^{(K)}}{\pi}\right)^{1/2} k^{-1/2}\, e^{\mp i\pi/4}\, c^{(K)}_{i\binom{1}{2}}. \tag{3.83}$$

On small scales, the spatial curvature perturbation [eqn. (3.41)], the fractional energy density perturbation [eqn. (3.49)], the longitudinal peculiar velocity perturbation [eqn. (3.51)] and the gauge-invariant fractional energy density perturbation [eqn. (3.76)] also oscillate; their amplitudes vary according to a^{-5}, a, a and a.

On large scales ($k \ll aH$) the scalar field perturbation, eqn. (3.20), is described by

$$\phi_K(k,x) = \left(\bar{c}_1\, k^{-5/2} + c_2^{(K)}\right) e^{-x} + \bar{c}_2\, k^{-1} + \bar{c}_3\, k\, e^{4x/3} + \bar{c}_4\, k\, e^{4x/3} \ln\left(\lambda_K\, e^{2x/3}\right) + \cdots, \tag{3.84}$$

where

$$\bar{c}_1 = -\frac{9}{2\sqrt{2}}\frac{1}{\pi}\Gamma^2\!\left(\frac{3}{4}\right)\left(\frac{\lambda_K}{k}\right)^{-5/2}\left(c_+^{(K)} e^{i\pi/4} + c_-^{(K)} e^{-i\pi/4}\right), \tag{3.85}$$

$$\bar{c}_2 = - i\,\frac{2}{\pi}\left(\frac{\lambda_K}{k}\right)^{-1}\left(c_+^{(K)} - c_-^{(K)}\right) \tag{3.86}$$

$$\bar{c}_3 = \frac{3}{14}\,\frac{\lambda_K}{k}\left[\left(c_+^{(K)} + c_-^{(K)}\right) + i\,\frac{2}{\pi}\left(\gamma - \ln(2) - \frac{11}{14}\right)\left(c_+^{(K)} - c_-^{(K)}\right)\right], \tag{3.87}$$

$$\bar{c}_4 = i\,\frac{3}{7\pi}\,\frac{\lambda_K}{k}\left(c_+^{(K)} - c_-^{(K)}\right), \tag{3.88}$$

where Γ is the gamma function and γ is Euler's constant. It is pleasing to note that the time dependence of the first three terms in eqn. (3.84) agrees with what was found in Sec. 3 of Ref. [8]. The term proportional to $\bar{c}_3$ is the dominant adiabatic solution while that proportional to $\bar{c}_2$ is a subdominant adiabatic solution. It is interesting to note that the large-scale asymptotic expansion contains terms that depend logarithmically on spatial momentum (these terms were not retained in the expansions of Sec. 3.3 of Ref. [1]).

On large scales the expressions for gravitational wave perturbations, eqns. (3.34), become

$$h_i^{(K)}(k,x) = \bar{c}_i^1 + \bar{c}_i^2 \ln\left(\lambda_K\, e^{2x/3}\right) + \bar{c}_i^3\, k^2\, e^{4x/3} + \bar{c}_i^4\, k^2\, e^{4x/3} \ln\left(\lambda_K\, e^{2x/3}\right) + \cdots, \tag{3.89}$$

where i is either $+$ or $\times$ and we have defined

$$\bar{c}_i^1 = \left(c_{i1}^{(K)} + c_{i2}^{(K)}\right) + i\,\frac{2}{\pi}\left(\gamma - \ln(2)\right)\left(c_{i1}^{(K)} - c_{i2}^{(K)}\right), \tag{3.90}$$

$$\bar{c}_i^2 = i\,\frac{2}{\pi}\left(c_{i1}^{(K)} - c_{i2}^{(K)}\right) \tag{3.91}$$

$$\bar{c}_i^3 = -\frac{1}{4}\left(\frac{\lambda_K}{k}\right)^2\left[\left(c_{i1}^{(K)} + c_{i2}^{(K)}\right) + i\,\frac{2}{\pi}\left(\gamma - \ln(2) - 1\right)\left(c_{i1}^{(K)} - c_{i2}^{(K)}\right)\right], \tag{3.92}$$

$$\bar{c}_i^4 = -\frac{i}{2\pi}\left(\frac{\lambda_K}{k}\right)^2\left(c_{i1}^{(K)} - c_{i2}^{(K)}\right). \tag{3.93}$$

We note that the leading large-scale gravitational wave perturbation [the term proportional to $\bar{c}_i^1$ in eqn. (3.89)] is time-independent.

The long wavelength part of the spatial curvature perturbation, eqn. (3.41), is given by

$${}^{(3)}R(k,x) = e^{-2x/3}\left[\left(\widetilde{c}_1\, k^{-1/2} + \widetilde{c}_2^{(K)}\, k^2\right) e^{-x} + \widetilde{c}_2\, k + \widetilde{c}_3\, k \ln\left(\lambda_K\, e^{2x/3}\right) + \cdots\right], \tag{3.94}$$

where we have defined

$$\widetilde{c}_1 = \frac{2}{\pi}\left(\frac{2}{3}\right)^{1/2}\Gamma^2\!\left(\frac{3}{4}\right)\left(\frac{\lambda_K}{k}\right)^{-1/2}\left(c_+^{(K)} e^{i\pi/4} + c_-^{(K)} e^{-i\pi/4}\right), \tag{3.95}$$

$$\widetilde{c}_2^{(K)} = -\frac{8}{9\sqrt{3}}\left(\frac{\lambda_K}{k}\right)^2 c_2^{(K)}, \tag{3.96}$$

$$\widetilde{c}_2 = \frac{4}{3\sqrt{3}}\frac{\lambda_K}{k}\left[-\left(c_+^{(K)} + c_-^{(K)}\right) - i\,\frac{2}{\pi}\left(\gamma - \ln(2) - \frac{2}{3}\right)\left(c_+^{(K)} - c_-^{(K)}\right)\right], \tag{3.97}$$

$$\widetilde{c}_3 = -\frac{i}{\pi}\frac{8}{3\sqrt{3}}\left(c_+^{(K)} - c_-^{(K)}\right). \tag{3.98}$$

Here, the term proportional to $\widetilde{c}_2$ is the dominant adiabatic solution. We note that the contribution of this solution to $a^{2\,(3)}R$ is time-independent — perturbations outside the Hubble radius in the curvature of the spatial hypersurface, corresponding to the dominant adiabatic solution, evolve in the expected fashion.

The fractional energy density perturbation, eqn. (3.49), has the large-scale expansion

$$\delta_{\Phi(K)}(k,x) = \left(\check{c}_1\,k^{-5/2} - \sqrt{3}\,c_2^{(K)}\right)e^{-x} + \check{c}_2\,k\,e^{4x/3} + \check{c}_3\,k\,e^{4x/3}\ln\left(\lambda_K\,e^{2x/3}\right) + \cdots, \tag{3.99}$$

where we have defined

$$\check{c}_1 = \frac{9}{2\pi}\left(\frac{3}{2}\right)^{1/2}\Gamma^2\!\left(\frac{3}{4}\right)\left(\frac{\lambda_K}{k}\right)^{-5/2}\left(c_+^{(K)}e^{i\pi/4} + c_-^{(K)}e^{-i\pi/4}\right), \tag{3.100}$$

$$\check{c}_2 = \frac{2}{7}\sqrt{3}\,\frac{\lambda_K}{k}\left[\left(c_+^{(K)} + c_-^{(K)}\right) + i\,\frac{2}{\pi}\left(\gamma - \ln(2) - \frac{2}{7}\right)\left(c_+^{(K)} - c_-^{(K)}\right)\right], \tag{3.101}$$

$$\check{c}_3 = \frac{i}{\pi}\,\frac{4\sqrt{3}}{7}\,\frac{\lambda_K}{k}\left(c_+^{(K)} - c_-^{(K)}\right). \tag{3.102}$$

The term proportional to $\check{c}_2$ in eqn. (3.99) is the dominant adiabatic solution. The longitudinal peculiar velocity perturbation, eqn. (3.51), has the large-scale form

$$v^1_{\Phi(K)}(k,x) = -\frac{i}{\sqrt{3}}\frac{\lambda_K}{k}\times \tag{3.103}$$
$$\left[\left(\bar{c}_1\,k^{-3/2} + c_2^{(K)}\,k\right)e^{-x/3} + \bar{c}_2\,e^{2x/3} + \bar{c}_3\,k^2\,e^{2x} + \bar{c}_4\,k^2\,e^{2x}\ln\left(\lambda_K\,e^{2x/3}\right) + \cdots\right],$$

where the constants $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$ and $\bar{c}_4$ have been defined in eqns. (3.85) – (3.88).

In the large-scale limit the gauge-invariant fractional energy density perturbation, eqn. (3.76), reduces to

$$\Delta_{\Phi(K)}(k,x) = \check{c}_1\,k^{-1} + \check{c}_2\,k\,e^{4x/3} + \check{c}_3\,k\,e^{4x/3}\ln\left(\lambda_K\,e^{2x/3}\right) + \cdots, \tag{3.104}$$

where we have defined

$$\breve{c}_1 = -\,i\,\frac{2\sqrt{3}}{\pi}\left(\frac{\lambda_K}{k}\right)^{-1}\left(c_+^{(K)} - c_-^{(K)}\right), \tag{3.105}$$

$$\breve{c}_2 = \frac{\sqrt{3}}{2}\,\frac{\lambda_K}{k}\left[\left(c_+^{(K)} + c_-^{(K)}\right) + i\,\frac{2}{\pi}\left(\gamma - \ln(2) - \frac{1}{2}\right)\left(c_+^{(K)} - c_-^{(K)}\right)\right], \tag{3.106}$$

$$\breve{c}_3 = i\,\frac{\sqrt{3}}{\pi}\,\frac{\lambda_K}{k}\left(c_+^{(K)} - c_-^{(K)}\right). \tag{3.107}$$

The term proportional to $\breve{c}_2$ in eqn. (3.104) is the dominant adiabatic solution; it has the same dependence on spatial momentum and on x as the dominant adiabatic solution in eqn. (3.99). It is interesting to note that the coefficients of the adiabatic terms in $\delta_{\Phi(K)}$ and $\Delta_{\Phi(K)}$, eqns. (3.101) and (3.106), differ; since, as we shall see in Sec. 6.1, the leading large-scale contributions to the constants of integration result in $c_-^{(K)} = c_+^{(K)}$, we find that the large-scale adiabatic contributions to the energy density two-point functions are related by $\left\langle \delta_{\Phi(K)}\,\delta^*_{\Phi(K)}\right\rangle = (16/49)\,\left\langle \Delta_{\Phi(K)}\,\Delta^*_{\Phi(K)}\right\rangle$.

4. Joining Conditions

The main technical advantage of the cosmological model considered here is that the transitions do not involve a change in the kind of matter, i.e., they are between two different scalar field dominated epochs (unlike, for instance, the "reheating" transition of Ref. [1] which is between a scalar field dominated epoch and a radiation dominated epoch). Furthermore, instead of the local energy density transition criterion used in Ref. [1], we assume that the transitions occur instantaneously, at different values of synchronous gauge time in different regions of space, when the scalar field reaches an appropriate value, i.e., we model a transition by replacing a pretransition scalar field potential by a posttransition scalar field potential at a hypersurface on which the scalar field is spatially homogeneous and has the relevant value. This transition spatial hypersurface, which is determined by the local (total) scalar field, differs from the corresponding synchronous gauge constant time hypersurface.

Aside from these changes, the procedure we use to determine joining conditions is similar to that described in Ref. [6]. In short, they are derived by requiring that the equations of motion remain singularity-free on the transition hypersurface. To condense the needed algebraic manipulations we derive joining conditions for a single transition "at" $\Phi = \Phi^{(1)}$ from a scalar field dominated epoch denoted I to a scalar field dominated epoch denoted R.

It proves convenient to define the following combinations of step functions,

$$\Theta_I \equiv 1 - \Theta\Big(\Phi - \Phi^{(1)}\Big)\,, \tag{4.1}$$

$$\Theta_R \equiv \Theta\Big(\Phi - \Phi^{(1)}\Big)\,, \tag{4.2}$$

where the Heaviside step function Θ is defined by eqn. (2.3). These functions form a mutually orthonormal set, i.e., they satisfy, for positive, integral n, m,

$$\Theta_I{}^n\, \Theta_R{}^m = 0, \tag{4.3}$$

$$\Theta_I{}^n = \Theta_I, \tag{4.4}$$

$$\Theta_R{}^n = \Theta_R. \tag{4.5}$$

4.1 Joining Conditions for Spatially Homogeneous Fields

The transition we consider is characterized by the background scalar field potential

$$V(\Phi_b) = V_I(\Phi_{bI})\,\Theta_I + V_R(\Phi_{bR})\,\Theta_R, \tag{4.6}$$

where the functions V_I and V_R may remain unspecified at this stage (we only require that they line up at the transition). In this subsection we consider the spatially homogeneous parts of the metric and scalar field and so may discard any spatial dependence in Θ_I and Θ_R, i.e., we may replace Φ by Φ_b in eqns. (4.1) and (4.2) (more precisely, the spatial hypersurface at which the joining is done differs from the corresponding synchronous gauge constant time hypersurface only by terms of first order in spatial irregularities, i.e., we may choose to join the spatially homogeneous parts of the fields at the synchronous gauge hypersurface).

The spatially homogeneous fields may be written

$$\Phi_b(t) = \Phi_{bI}(t)\,\Theta_I + \Phi_{bR}(t)\,\Theta_R, \tag{4.7}$$

$$a(t) = a_I(t)\,\Theta_I + a_R(t)\,\Theta_R, \tag{4.8}$$

where we require (to ensure that the field-space coordinates do not jump at a transition)

$$\Phi_{bR}(t_1) = \Phi_{bI}(t_1), \tag{4.9}$$

$$a_R(t_1) = a_I(t_1), \tag{4.10}$$

where t_1 is the value of synchronous gauge time at the transition spatial hypersurface. The temporal derivative of a step function may be defined as

$$\frac{d}{dt}\Theta\Big(\Phi_b - \Phi^{(1)}\Big) = \frac{1}{2}\left(\dot{\Phi}_{bI}(t_1) + \dot{\Phi}_{bR}(t_1)\right)\delta\Big(\Phi_b - \Phi^{(1)}\Big)\,, \tag{4.11}$$

where δ is the Dirac delta function which, in what follows, we abbreviate as

$$\delta_{(1)} \equiv \delta\Big(\Phi_b - \Phi^{(1)}\Big)\,. \tag{4.12}$$

Using eqns. (4.1), (4.2) and (4.11) we find, from eqn. (4.7),

$$\dot{\Phi}_b(t) = \dot{\Phi}_{bI}(t)\,\Theta_I + \dot{\Phi}_{bR}(t)\,\Theta_R - \frac{1}{2}\left(\Phi_{bI}(t) - \Phi_{bR}(t)\right)\left(\dot{\Phi}_{bI}(t_1) + \dot{\Phi}_{bR}(t_1)\right)\delta_{(1)}. \tag{4.13}$$

Now, because of the Dirac delta function, the first scalar field dependent factor in the last term on the right hand side of this equation may be evaluated at $t = t_1$ (since $\Phi_b = \Phi^{(1)}$ only at $t = t_1$); from eqn. (4.9) we see that it vanishes, so

$$\dot{\Phi}_b(t) = \dot{\Phi}_{bI}(t)\,\Theta_I + \dot{\Phi}_{bR}(t)\,\Theta_R. \tag{4.14}$$

In a similar fashion, eqns. (4.8) and (4.10) result in

$$\dot{a}(t) = \dot{a}_I(t)\,\Theta_I + \dot{a}_R(t)\,\Theta_R \tag{4.15}$$

and the definitions of Θ_I and Θ_R ensure

$$\frac{\dot{a}(t)}{a(t)} = \frac{\dot{a}_I(t)}{a_I(t)}\,\Theta_I + \frac{\dot{a}_R(t)}{a_R(t)}\,\Theta_R. \tag{4.16}$$

To derive joining conditions we require that the equations of motion, eqns. (3.3) – (3.5), remain singularity-free at the transition. It is straightforward to establish that eqn. (3.4) is nonsingular at the transition and so does not result in a joining condition. More specifically, from eqns. (4.3) – (4.5) and (4.16) we find that the left hand side of eqn. (3.4) is

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{a}_I}{a_I}\right)^2 \Theta_I + \left(\frac{\dot{a}_R}{a_R}\right)^2 \Theta_R, \tag{4.17}$$

while from eqns. (4.3) – (4.6) and (4.14) the right hand side of eqn. (3.4) is given by

$$\frac{1}{12}\left[(\dot{\Phi}_b)^2 + V(\Phi_b)\right] = \frac{1}{12}\left[(\dot{\Phi}_{bI})^2 + V_I(\Phi_{bI})\right]\Theta_I + \frac{1}{12}\left[(\dot{\Phi}_{bR})^2 + V_R(\Phi_{bR})\right]\Theta_R. \tag{4.18}$$

All that results from combining eqns. (3.4), (4.17) and (4.18) are a part of the set of equations of motion for the background fields in the two epochs. Eqn. (3.5) may be reexpressed as

$$\ddot{a} = -\frac{1}{12}\left[2(\dot{\Phi}_b)^2 - V(\Phi_b)\right] a. \tag{4.19}$$

From eqns. (4.11), (4.12) and (4.15) we find

$$\ddot{a}(t) = \ddot{a}_I(t)\,\Theta_I + \ddot{a}_R(t)\,\Theta_R - \frac{1}{2}\left(\dot{a}_I(t) - \dot{a}_R(t)\right)\left(\dot{\Phi}_{bI}(t_1) + \dot{\Phi}_{bR}(t_1)\right)\delta_{(1)}, \tag{4.20}$$

while the right hand side of eqn. (4.19) is given by

$$-\frac{1}{12}\left[2(\dot{\Phi}_{bI})^2 - V_I(\Phi_{bI})\right] a_I\,\Theta_I - \frac{1}{12}\left[2(\dot{\Phi}_{bR})^2 - V_R(\Phi_{bR})\right] a_R\,\Theta_R. \tag{4.21}$$

Making use of the Dirac delta function in eqn. (4.20), we see that eqn. (4.19) is singularity-free at the transition provided

$$\dot{a}_R(t_1) = \dot{a}_I(t_1) \tag{4.22}$$

(we have assumed that $\dot{\Phi}_b$ does not change sign at the transition). We note that this equation, along with eqns. (4.10) and (4.16), implies that the Hubble parameter does not jump at the transition. We must now examine eqn. (3.3). Using the fact that the Dirac delta function is the derivative of the Heaviside step function we find, from eqn. (4.6),

$$V'(\Phi_b) = V_I'(\Phi_{bI})\,\Theta_I + V_R'(\Phi_{bR})\,\Theta_R - \left(V_I(\Phi_{bI}) - V_R(\Phi_{bR})\right)\delta_{(1)}. \tag{4.23}$$

Since we have required that the two parts of the potential match at the transition we may drop the last term. With techniques similar to those used above, eqn. (3.3) reduces to

$$\left[\ddot{\Phi}_{bI} + 3\,\frac{\dot{a}_I}{a_I}\,\dot{\Phi}_{bI} + \frac{1}{2}V_I'(\Phi_{bI})\right]\Theta_I + \left[\ddot{\Phi}_{bR} + 3\,\frac{\dot{a}_R}{a_R}\,\dot{\Phi}_{bR} + \frac{1}{2}V_R'(\Phi_{bR})\right]\Theta_R$$
$$-\frac{1}{2}\left[\left(\dot{\Phi}_{bI}(t_1)\right)^2 - \left(\dot{\Phi}_{bR}(t_1)\right)^2\right]\delta_{(1)} = 0, \tag{4.24}$$

which results in the joining condition

$$\dot{\Phi}_{bR}(t_1) = \dot{\Phi}_{bI}(t_1) \tag{4.25}$$

(the physics of the problem requires that we discard the other possible joining condition).

The energy density and pressure of the spatially homogeneous part of the scalar field, eqns. (3.42) and (3.43), become

$$\rho_{b\Phi} = \frac{m_p^2}{32\pi}\left[(\dot{\Phi}_{bI})^2 + V_I(\Phi_{bI})\right]\Theta_I + \frac{m_p^2}{32\pi}\left[(\dot{\Phi}_{bR})^2 + V_R(\Phi_{bR})\right]\Theta_R, \tag{4.26}$$
$$p_{b\Phi} = \frac{m_p^2}{32\pi}\left[(\dot{\Phi}_{bI})^2 - V_I(\Phi_{bI})\right]\Theta_I + \frac{m_p^2}{32\pi}\left[(\dot{\Phi}_{bR})^2 - V_R(\Phi_{bR})\right]\Theta_R. \tag{4.27}$$

We note that eqn. (4.25) (with the matching of the parts of the potential at the transition) ensures that the background energy density and pressure do not jump at the transition.

4.2 Joining Conditions for Spatial Irregularities

We have chosen to define the transition hypersurface as a hypersurface on which the local scalar field is spatially homogeneous and has the appropriate value. Assuming that the transition hypersurface is "close" to the $t = t_1$ synchronous gauge constant time hypersurface (i.e., that they only differ by terms of first order in spatial irregularity) we may utilize the time-independent space-dependent time translation gauge freedom (in synchronous gauge) to make a (first order in spatial irregularity) coordinate change to remove the spatially inhomogeneous part of the scalar field. Consider the coordinates $\bar{x}^\mu = (\bar{t}, \bar{x}^i)$ defined by

$$\bar{t} = t - \Delta t(t_1, \vec{x}), \tag{4.28}$$

$$\bar{x}^i = x^i - f^i(t, \vec{x}), \tag{4.29}$$

where Δt and f^i are of first order in spatial irregularity and will be determined by requiring that the constant $\bar{t}_1 (= t_1 - \Delta t)$ spatial hypersurface is a synchronous and instantaneously spatially homogeneous local scalar field spatial hypersurface, i.e., by requiring

$$\left[\frac{\partial \bar{\Phi}}{\partial \bar{x}^i}(\bar{x})\right]\bigg|_{\bar{t}=\bar{t}_1} = 0. \tag{4.30}$$

Utilizing the fact that the local (total) scalar field is a general coordinate scalar [i.e., $\bar{\Phi}(\bar{x}) = \Phi(x)$], it may be shown that to first order in spatial irregularity eqn. (4.30) reduces to

$$\frac{\partial}{\partial x^i}\left[\phi(t_1, \vec{x}) + \dot{\Phi}_b(t_1)\Delta t(t_1, \vec{x})\right] = 0, \tag{4.31}$$

or (since the space-independent constant of integration must vanish)

$$\Delta t(t_1, \vec{x}) = -\frac{\phi(t_1, \vec{x})}{\dot{\Phi}_b(t_1)}. \tag{4.32}$$

Expanding the scalar field, in the $\bar{x}$ coordinates, about a spatially homogeneous part,

$$\bar{\Phi}(\bar{x}) = \bar{\Phi}_b(\bar{t}) + \bar{\phi}(\bar{x}); \tag{4.33}$$

it is straightforward to establish that the coordinate change, eqns. (4.28) and (4.29), relates the perturbation in the barred coordinate system to that in the unbarred system through

$$\bar{\phi}(\bar{x}) = \phi(x) + \dot{\Phi}_b(t)\Delta t(t_1, \vec{x}). \tag{4.34}$$

Clearly [eqn. (4.32)]

$$\bar{\phi}(\bar{t}_1, \bar{x}^i) = 0, \tag{4.35}$$

i.e., the spatially inhomogeneous part of the scalar field vanishes on the transition spatial hypersurface (in the barred coordinates).

If one requires that the barred coordinates be synchronous then f^i, eqn. (4.29), is

$$f^i(t,\vec{x}) = \partial_i \Delta t(t_1,\vec{x}) \int^t \frac{dt'}{a^2(t')}, \tag{4.36}$$

where we have ignored time-independent space-dependent spatial reparametrizations (see the discussion around eqns. (3.13) – (3.15) of Ref. [6]). With this choice for f^i one may expand, to first order in spatial irregularity,

$$\bar{g}_{00}(\bar{x}) = 1, \tag{4.37}$$

$$\bar{g}_{0i}(\bar{x}) = 0, \tag{4.38}$$

$$\bar{g}_{ij}(\bar{x}) = -\bar{a}^2(\bar{t}) \left[\delta_{ij} - \bar{h}_{ij}(\bar{x})\right], \tag{4.39}$$

where the infinitesimal coordinate change [eqns. (4.28), (4.29) and (4.36)] relate

$$\bar{h}_{ij}(\bar{x}) = h_{ij}(x) - 2\partial_i\partial_j \Delta t(t_1,\vec{x}) \int^t \frac{dt'}{a^2(t')} - 2\,\frac{\dot{a}(t)}{a(t)}\,\Delta t(t_1,\vec{x})\,\delta_{ij}. \tag{4.40}$$

Before examining the transition spatial hypersurface singularity structure of the equations of motion for the perturbations we consider the form the fields assume. We may write

$$\bar{\Phi}(\bar{x}) = \bar{\Phi}_I(\bar{x})\,\bar{\Theta}_I + \bar{\Phi}_R(\bar{x})\,\bar{\Theta}_R, \tag{4.41}$$

$$\bar{g}_{ij}(\bar{x}) = \bar{g}^{(I)}_{ij}(\bar{x})\,\bar{\Theta}_I + \bar{g}^{(R)}_{ij}(\bar{x})\,\bar{\Theta}_R, \tag{4.42}$$

where $\bar{\Theta}_I$ and $\bar{\Theta}_R$ are just Θ_I and Θ_R [eqns. (4.1) and (4.2)] with Φ replaced by $\bar{\Phi}$. Using eqn. (4.33) and Taylor expanding the step functions, eqn. (4.41) may be reexpressed, to lowest order, as

$$\bar{\Phi}_b(\bar{t}) = \bar{\Phi}_{bI}(\bar{t})\,\bar{\Theta}_{bI} + \bar{\Phi}_{bR}(\bar{t})\,\bar{\Theta}_{bR} \tag{4.43}$$

(here $\bar{\Theta}_{bI}$ and $\bar{\Theta}_{bR}$ are just Θ_I and Θ_R with Φ replaced by $\bar{\Phi}_b$), which (to lowest order) coincides with eqn. (4.7), and to linear order

$$\bar{\phi}(\bar{x}) = \bar{\phi}_I(\bar{x})\,\bar{\Theta}_{bI} + \bar{\phi}_R(\bar{x})\,\bar{\Theta}_{bR} - \Big(\bar{\Phi}_{bI}(\bar{t})\bar{\phi}_I(\bar{x}) - \bar{\Phi}_{bR}(\bar{t})\bar{\phi}_R(\bar{x})\Big)\,\bar{\delta}_{(1)}, \tag{4.44}$$

where we have abbreviated the Dirac delta function

$$\bar{\delta}_{(1)} = \delta\Big(\bar{\Phi}_b - \Phi^{(1)}\Big) \tag{4.45}$$

(and not bothered to symmetrize the last term on the right hand side of eqn. (4.44) with respect to the perturbations). Making use of the Dirac delta function we may rewrite the last term in eqn. (4.44), to lowest order, as

$$-\Big(\bar{\Phi}_{bI}(\bar{t}_1)\bar{\phi}_I(\bar{t}_1,\bar{x}^i) - \bar{\Phi}_{bR}(\bar{t}_1)\bar{\phi}_R(\bar{t}_1,\bar{x}^i)\Big)\,\bar{\delta}_{(1)}, \tag{4.46}$$

which vanishes because of eqn. (4.35), so we have

$$\bar{\phi}(\bar{x}) = \bar{\phi}_I(\bar{x})\,\bar{\Theta}_{bI} + \bar{\phi}_R(\bar{x})\,\bar{\Theta}_{bR}. \tag{4.47}$$

In a similar fashion, we find, from eqn. (4.42),

$$\bar{a}(\bar{t}) = \bar{a}_I(\bar{t})\,\bar{\Theta}_{bI} + \bar{a}_R(\bar{t})\,\bar{\Theta}_{bR}, \tag{4.48}$$

$$\bar{h}_{ij}(\bar{x}) = \bar{h}^{(I)}_{ij}(\bar{x})\,\bar{\Theta}_{bI} + \bar{h}^{(R)}_{ij}(\bar{x})\,\bar{\Theta}_{bR}. \tag{4.49}$$

In the barred coordinates, with the expansions of eqns. (4.33) and (4.37) – (4.39), we find that the equations of motion for spatial irregularities are

$$\bar{\partial}_0^2\bar{\phi} + 3\,\frac{\bar{\partial}_0\bar{a}}{\bar{a}}\,\bar{\partial}_0\bar{\phi} - \frac{1}{\bar{a}^2}\bar{\partial}_1^2\bar{\phi} + \frac{1}{2}V''(\bar{\Phi}_b)\bar{\phi} = \frac{1}{2}\bar{\partial}_0\bar{\Phi}_b\bar{\partial}_0\bar{h}, \tag{4.50}$$

$$\bar{\partial}_0^2\bar{h} + 2\,\frac{\bar{\partial}_0\bar{a}}{\bar{a}}\,\bar{\partial}_0\bar{h} = 2\bar{\partial}_0\bar{\Phi}_b\bar{\partial}_0\bar{\phi} - \frac{1}{2}V'(\bar{\Phi}_b)\bar{\phi}, \tag{4.51}$$

$$\bar{\partial}_0\bar{\partial}_1\bar{h}_{1I} = 0, \tag{4.52}$$

$$\bar{\partial}_0^2\bar{h}_{I1} + 3\,\frac{\bar{\partial}_0\bar{a}}{\bar{a}}\,\bar{\partial}_0\bar{h}_{I1} = 0, \tag{4.53}$$

$$\bar{\partial}_0^2\bar{h}_i + 3\,\frac{\bar{\partial}_0\bar{a}}{\bar{a}}\,\bar{\partial}_0\bar{h}_i - \frac{1}{\bar{a}^2}\bar{\partial}_1^2\bar{h}_i = 0, \tag{4.54}$$

$$\bar{\partial}_0\bar{\partial}_1(\bar{h}_{22} + \bar{h}_{33}) = \bar{\partial}_0\bar{\Phi}_b\bar{\partial}_1\bar{\phi}, \tag{4.55}$$

$$\bar{\partial}_0^2(\bar{h}_{22}+\bar{h}_{33}) + 3\,\frac{\bar{\partial}_0\bar{a}}{\bar{a}}\,\bar{\partial}_0(\bar{h}_{22}+\bar{h}_{33}) = \bar{\partial}_0\bar{\Phi}_b\bar{\partial}_0\bar{\phi} - \frac{1}{2}V'(\bar{\Phi}_b)\bar{\phi}, \tag{4.56}$$

$$\frac{1}{\bar{a}^2}\bar{\partial}_1^2(\bar{h}_{22}+\bar{h}_{33}) = 2\,\frac{\bar{\partial}_0\bar{a}}{\bar{a}}\,\bar{\partial}_0\bar{h} + \bar{\partial}_0\bar{\Phi}_b\bar{\partial}_0\bar{\phi} + \frac{1}{2}V'(\bar{\Phi}_b)\bar{\phi}, \tag{4.57}$$

where the various symbols have their usual meaning (in particular, in eqn. (4.54) i is either $+$ or $\times$) and we have chosen to focus on a plane wave with propagation vector along the $\bar{x}^1$ direction. In these equations the background scalar field potential is

$$V(\bar{\Phi}_b) = V_I(\bar{\Phi}_{bI})\,\bar{\Theta}_{bI} + V_R(\bar{\Phi}_{bR})\,\bar{\Theta}_{bR}. \tag{4.58}$$

To determine joining conditions for the spatial irregularities we require that equations (4.50) – (4.57) remain singularity-free at the transition. The general procedure is similar to that used in Sec. 4.1 and so we can be brief; two useful relations, not needed there, are

$$\bar{\Theta}_{bI}\,\bar{\delta}_{(1)} = \frac{1}{2}\bar{\delta}_{(1)}, \tag{4.59}$$

$$\bar{\Theta}_{bR}\,\bar{\delta}_{(1)} = \frac{1}{2}\bar{\delta}_{(1)}. \tag{4.60}$$

To illustrate the general procedure we derive the joining conditions that follow from eqns. (4.50) and (4.51) and then tabulate those that result from eqns. (4.52) – (4.57). Eqn. (4.50) may be reexpressed as

$$\begin{aligned}
&\left[\bar{\partial}_0^2\bar{\phi}_I + 3\,\frac{\bar{\partial}_0\bar{a}_I}{\bar{a}_I}\,\bar{\partial}_0\bar{\phi}_I - \frac{1}{\bar{a}_I^2}\bar{\partial}_1^2\bar{\phi}_I + \frac{1}{2}V_I''(\bar{\Phi}_{bI})\bar{\phi}_I - \frac{1}{2}\bar{\partial}_0\bar{\Phi}_{bI}\bar{\partial}_0\bar{h}^{(I)}\right]\bar{\Theta}_{bI} \\
&+\left[\bar{\partial}_0^2\bar{\phi}_R + 3\,\frac{\bar{\partial}_0\bar{a}_R}{\bar{a}_R}\,\bar{\partial}_0\bar{\phi}_R - \frac{1}{\bar{a}_R^2}\bar{\partial}_1^2\bar{\phi}_R + \frac{1}{2}V_R''(\bar{\Phi}_{bR})\bar{\phi}_R - \frac{1}{2}\bar{\partial}_0\bar{\Phi}_{bR}\bar{\partial}_0\bar{h}^{(R)}\right]\bar{\Theta}_{bR} \\
&-\frac{1}{2}\Bigg[\left(\bar{\partial}_0\bar{\phi}_I - \bar{\partial}_0\bar{\phi}_R\right)\left(\bar{\partial}_0\bar{\Phi}_{bI}(\bar{t}_1) + \bar{\partial}_0\bar{\Phi}_{bR}(\bar{t}_1)\right) + \frac{1}{2}\left(\bar{\phi}_I + \bar{\phi}_R\right)\left(V_I'(\bar{\Phi}_{bI}) - V_R'(\bar{\Phi}_{bR})\right) \\
&\qquad - \frac{1}{4}\left(\bar{h}^{(I)} - \bar{h}^{(R)}\right)\left(\bar{\partial}_0\bar{\Phi}_{bI} + \bar{\partial}_0\bar{\Phi}_{bR}\right)\left(\bar{\partial}_0\bar{\Phi}_{bI}(\bar{t}_1) + \bar{\partial}_0\bar{\Phi}_{bR}(\bar{t}_1)\right)\Bigg]\bar{\delta}_{(1)} = 0,
\end{aligned} \tag{4.61}$$

where we have made use of the fact that the potential does not jump at the transition and the time argument of all variables for which it is unspecified is $\bar{t}$. In a similar fashion eqn. (4.51) may be rewritten as

$$\left[\bar{\partial}_0^2\bar{h}^{(I)} + 2\,\frac{\bar{\partial}_0\bar{a}_I}{\bar{a}_I}\,\bar{\partial}_0\bar{h}^{(I)} - 2\bar{\partial}_0\bar{\Phi}_{bI}\bar{\partial}_0\bar{\phi}_I + \frac{1}{2}V_I'(\bar{\Phi}_{bI})\bar{\phi}_I\right]\bar{\Theta}_{bI}$$

$$+\left[\bar\partial_0^2\bar h^{(R)}+2\,\frac{\bar\partial_0\bar a_R}{\bar a_R}\,\bar\partial_0\bar h^{(R)}-2\bar\partial_0\bar\Phi_{bR}\bar\partial_0\bar\phi_R+\frac{1}{2}V_R'(\bar\Phi_{bR})\bar\phi_R\right]\bar\Theta_{bR}$$
$$-\left[\bar\partial_0\bar h^{(I)}-\bar\partial_0\bar h^{(R)}+\frac{1}{2}\left(\bar h^{(I)}-\bar h^{(R)}\right)\left(\frac{\bar\partial_0\bar a_I}{\bar a_I}+\frac{\bar\partial_0\bar a_R}{\bar a_R}\right)\right]\left(\bar\partial_0\bar\Phi_{bI}(\bar t_1)+\bar\partial_0\bar\Phi_{bR}(\bar t_1)\right)\;\bar\delta_{(1)}$$
$$-\frac{1}{4}\left(\bar h^{(I)}-\bar h^{(R)}\right)\left(\bar\partial_0\bar\Phi_{bI}(\bar t_1)+\bar\partial_0\bar\Phi_{bR}(\bar t_1)\right)^2\bar\delta_{(1)}'=0, \tag{4.62}$$

where $\bar\delta_{(1)}'$ represents the derivative of the delta function. We now require that the coefficients of the singular terms in eqns. (4.61) and (4.62) vanish. (The coefficients of the linear combination of step functions in these equations are some of the equations of motion.) The derivative of the delta function in eqn. (4.62) is nonzero only when $\bar t=\bar t_1$, so we require

$$\bar h^{(R)}(\bar t_1)=\bar h^{(I)}(\bar t_1). \tag{4.63}$$

Similarly, from the coefficient of the Dirac delta function in eqn. (4.62) we get

$$\bar\partial_0\bar h^{(R)}(\bar t_1)=\bar\partial_0\bar h^{(I)}(\bar t_1) \tag{4.64}$$

(since, from eqn. (4.63), the first field-dependent factor in the last term in the square brackets in the coefficient of $\bar\delta_{(1)}$ in eqn. (4.62) vanishes at $\bar t=\bar t_1$). $\bar\phi_I$ and $\bar\phi_R$ vanish at $\bar t=\bar t_1$ and using eqn. (4.63) the coefficient of the Dirac delta function in eqn. (4.61) vanishes provided

$$\bar\partial_0\bar\phi_R(\bar t_1)=\bar\partial_0\bar\phi_I(\bar t_1). \tag{4.65}$$

In a similar manner we find that eqns. (4.52) – (4.57) result in

$$\bar h_{I1}^{(R)}(\bar t_1)=\bar h_{I1}^{(I)}(\bar t_1), \tag{4.66}$$
$$\bar\partial_0\bar h_{I1}^{(R)}(\bar t_1)=\bar\partial_0\bar h_{I1}^{(I)}(\bar t_1), \tag{4.67}$$
$$\bar h_i^{(R)}(\bar t_1)=\bar h_i^{(I)}(\bar t_1), \tag{4.68}$$
$$\bar\partial_0\bar h_i^{(R)}(\bar t_1)=\bar\partial_0\bar h_i^{(I)}(\bar t_1), \tag{4.69}$$
$$\bar h_{22}^{(R)}(\bar t_1)+\bar h_{33}^{(R)}(\bar t_1)=\bar h_{22}^{(I)}(\bar t_1)+\bar h_{33}^{(I)}(\bar t_1), \tag{4.70}$$
$$\bar\partial_0\left(\bar h_{22}^{(R)}+\bar h_{33}^{(R)}\right)(\bar t_1)=\bar\partial_0\left(\bar h_{22}^{(I)}+\bar h_{33}^{(I)}\right)(\bar t_1). \tag{4.71}$$

Not all of eqns. (4.63) – (4.71) are joining conditions — some of them are consistency conditions. It is interesting to note that all the metric perturbation joining and consistency conditions in the barred coordinate system, eqns. (4.63), (4.64) and (4.66) – (4.71), coincide with those found in Sec. 3 of Ref. [6].

It proves convenient to reexpress the joining and consistency conditions in terms of variables in the unbarred coordinate system. (The general procedure has been described in Ref. [6] so we can be brief.) Since the transition hypersurface cannot depend on whether we define it in the I dominated epoch or in the R dominated epoch, eqn. (4.32) implies

$$\phi_R(t_1,\vec{x}) = \phi_I(t_1,\vec{x}) \tag{4.72}$$

[where we have used eqn. (4.25)]. Using eqns. (3.5), (4.10), (4.22), (4.25), (4.32), (4.34), (4.40) and the relation (valid to lowest order)

$$\bar{\partial}_0 = \partial_0, \tag{4.73}$$

we find that eqns. (4.63) – (4.71) become

$$\dot{\phi}_R(t_1) = \dot{\phi}_I(t_1) - \frac{\phi_I(t_1)}{\dot{\Phi}_{bI}(t_1)}\left[\ddot{\Phi}_{bI}(t_1) - \ddot{\Phi}_{bR}(t_1)\right], \tag{4.74}$$

$$h^{(R)}(t_1) = h^{(I)}(t_1) - 2k^2\frac{\phi_I(t_1)}{\dot{\Phi}_{bI}(t_1)}\int^{t_1}\left[\frac{1}{a_I^2(t')} - \frac{1}{a_R^2(t')}\right]dt', \tag{4.75}$$

$$\dot{h}^{(R)}(t_1) = \dot{h}^{(I)}(t_1), \tag{4.76}$$

$$h_{I1}^{(R)}(t_1) = h_{I1}^{(I)}(t_1), \tag{4.77}$$

$$\dot{h}_{I1}^{(R)}(t_1) = \dot{h}_{I1}^{(I)}(t_1), \tag{4.78}$$

$$h_i^{(R)}(t_1) = h_i^{(I)}(t_1), \tag{4.79}$$

$$\dot{h}_i^{(R)}(t_1) = \dot{h}_i^{(I)}(t_1), \tag{4.80}$$

$$h_{22}^{(R)}(t_1) + h_{33}^{(R)}(t_1) = h_{22}^{(I)}(t_1) + h_{33}^{(I)}(t_1), \tag{4.81}$$

$$\dot{h}_{22}^{(R)}(t_1) + \dot{h}_{33}^{(R)}(t_1) = \dot{h}_{22}^{(I)}(t_1) + \dot{h}_{33}^{(I)}(t_1). \tag{4.82}$$

It is interesting to note that eqns. (4.77) – (4.81) coincide with eqns. (3.51), (3.57), (3.66), (3.67) and (3.53) of Ref. [6].

It is instructive to compare the number of posttransition constants that have to be determined with the number of equations we have derived. The constants c_2 and $c_\pm$ (which correspond to the time translation solution and combinations of the dominant and subdominant "adiabatic" solutions) are determined from eqns. (4.72), (4.74) and (4.81), while c_1 is fixed by eqn. (4.75). The constants in the gravitational wave solutions, c_{i1} and c_{i2}, are fixed by eqns. (4.79) and (4.80), while $c_{I,1}$ are determined from eqns. (4.77). This leaves three consistency conditions, eqns. (4.76), (4.78) and (4.82), which must be identically satisfied (provided, of course, that the equations of motion and the joining conditions are satisfied).

We now establish that the consistency conditions are identities. One of the (spatial momentum space) equations of motion for the h_{I1} component of the metric perturbation, in a scalar field dominated cosmological model, is

$$\dot{h}_{I1} = 0, \tag{4.83}$$

which implies the consistency condition (4.78). One of the equations of motion for the combination of metric perturbations related to the perturbation in space curvature is

$$\dot{h}_{22} + \dot{h}_{33} - \dot{\Phi}_b \phi = 0, \tag{4.84}$$

which [given eqns. (4.25) and (4.72)] implies the consistency condition (4.82). We now turn to eqn. (4.76) — the spatial momentum space equation of motion of interest is

$$-\frac{k^2}{a^2}\,(h_{22} + h_{33}) = 2\,\frac{\dot{a}}{a}\,\dot{h} + \dot{\Phi}_b \dot{\phi} + \frac{1}{2} V'(\Phi_b)\phi. \tag{4.85}$$

Using eqn. (3.3) to eliminate the derivative of the potential and using eqns. (4.10), (4.22), (4.25), (4.72), (4.74) and (4.81), one may show that this equation implies eqn. (4.76).

4.3 Joining Conditions for Derived Fields and Gauge-Invariant Joining Conditions

In this section we reexpress the scalar field joining conditions of the previous section as joining conditions for derived fields which describe perturbations in the equivalent spacetime-dependent "speed of sound" fluid model (Sec. 2 of Ref. [6]). Combinations of these gauge-dependent joining conditions are joining conditions for gauge-invariant variables (Sec. 4 of Ref. [6]) — in this section we also tabulate the relevant gauge-invariant joining conditions.

The spatially homogeneous part of the scalar field energy density and pressure do not jump at the transition (Sec. 4.1) — as a result the transitions in this model are gentler than those in the models of Refs. [1,3]. Operational definitions of the scalar field fractional energy density irregularity, peculiar velocity inhomogeneity and effective "speed of sound" are given in eqns. (3.45) – (3.47) above. We shall illustrate the derivation of the joining condition for the peculiar velocity irregularity and then tabulate those for the other effective fluid variables. From eqns. (3.46), (4.8), (4.14), (4.34) and (4.47) we have

$$\begin{aligned}(\rho_{b\Phi} + p_{b\Phi})\, v^i_{\Phi} &= \left(\rho_{b\Phi(I)} + p_{b\Phi(I)}\right) v^i_{\Phi(I)}\, \Theta_{bI} + \left(\rho_{b\Phi(R)} + p_{b\Phi(R)}\right) v^i_{\Phi(R)}\, \Theta_{bR} \\ &= -\frac{m_p^2}{16\pi}\left[\frac{\dot\Phi_{bI}}{a_I}\,\partial_i\phi_I\,\Theta_{bI} + \frac{\dot\Phi_{bR}}{a_R}\,\partial_i\phi_R\,\Theta_{bR}\right], \end{aligned} \tag{4.86}$$

where Θ_{bI} and Θ_{bR} are Θ_I and Θ_R with Φ replaced by Φ_b. Using eqns. (4.10), (4.25) and (4.72) as well as the fact that the spatially homogeneous part of the scalar field energy density and pressure do not jump at the transition, we have, at the transition,

$$v^i_{\Phi(R)}(t_1) = v^i_{\Phi(I)}(t_1). \tag{4.87}$$

This equation should be compared to the corresponding joining condition (eqn. (3.39) of Ref. [6]) in the case when the transition is modelled as occurring on a spatially homogeneous energy density hypersurface (instead of a spatially homogeneous scalar field hypersurface) — in the scalar field case the background energy density and pressure do not jump at the transition hypersurface and it is interesting to note that if one accounts for this in eqn. (3.39) of Ref. [6] it reduces to eqn. (4.87) above (we emphasize, however, that the transition prescriptions used here and in Ref. [6] are physically distinct). In a similar fashion one finds that the fractional energy density perturbation is given by

$$\delta_\Phi = \delta_{\Phi(I)}\,\Theta_{bI} + \delta_{\Phi(R)}\,\Theta_{bR}, \tag{4.88}$$

and that at the transition it obeys

$$\delta_{\Phi(R)}(t_1) = \delta_{\Phi(I)}(t_1). \tag{4.89}$$

Eqn. (3.10) of Ref. [6] reduces to this equation when the spatially homogeneous part of the energy density and pressure do not jump at the transition. It may also be shown that

$$c^2_{s\Phi}\,\rho_{b\Phi}\,\delta_\Phi = c^2_{s\Phi(I)}\,\rho_{b\Phi(I)}\,\delta_{\Phi(I)}\,\Theta_{bI} + c^2_{s\Phi(R)}\,\rho_{b\Phi(R)}\,\delta_{\Phi(R)}\,\Theta_{bR}; \tag{4.90}$$

in this case, however, since the effective "speed of sound" is related to a derivative of the pressure, we find that the relevant variable jumps at a transition,

$$\begin{aligned}\left[c^2_{s\Phi(R)}\,\rho_{b\Phi(R)}\,\delta_{\Phi(R)}\right](t_1) = &\left[c^2_{s\Phi(I)}\,\rho_{b\Phi(I)}\,\delta_{\Phi(I)}\right](t_1)\\ &+\frac{m_p^2}{16\pi}\left[V_I'\left(\Phi_{bI}(t_1)\right) - V_R'\left(\Phi_{bR}(t_1)\right)\right]\phi_I(t_1)\end{aligned} \tag{4.91}$$

(this is, of course, not unexpected).

The joining conditions for the perturbations may be reexpressed as joining conditions for gauge-invariant variables. For the gauge-invariant variables Δ_Φ and A_Φ (defined in eqns. (3.74) and (3.75) above) we find

$$\Delta_{\Phi(R)}(t_1) = \Delta_{\Phi(I)}(t_1), \tag{4.92}$$

$$A_{\Phi(R)}(t_1) = A_{\Phi(I)}(t_1). \tag{4.93}$$

Eqn. (4.92) coincides with eqn. (4.4) of Ref. [6] (for the spatially homogeneous energy density transition model) and eqn. (3.20) of Ref. [3] reduces to eqn. (4.93) above provided one allows for the matching of the background energy density and pressure in the spatially homogeneous scalar field transition hypersurface model. We again emphasize that the spatially homogeneous energy density transition prescription and the spatially homogeneous scalar field transition prescription are physically distinct models for a transition.

5. Joining

We have derived solutions of the relativistic linear perturbation equations in each of the epochs of the model (for inflation in Secs. 2 and 3 of Ref. [1], during "reheating" in Sec. 2 of Ref. [13] and for the kinetic energy dominated epoch in Sec. 3 above). The constants of integration in the inflation epoch solutions are determined from the quantum mechanical initial conditions of Refs. [4,1], while those in the "reheating" and kinetic energy dominated epoch solutions will be determined from the joining conditions of the previous section.

5.1 Joining the Spatially Homogeneous Fields

We first tabulate the solutions for the scale factor and the spatially homogeneous part of the scalar field in each of the epochs. During inflation we have (eqn. (2.5) of Ref. [1])

$$a_I(t) = a_0 \left[1 + M(t - t_0)\right]^{2/q}, \tag{5.1}$$

$$\Phi_{bI}(t) = \Phi^{(0)} + \sqrt{\frac{8}{q}} \ln\left[1 + M(t - t_0)\right], \tag{5.2}$$

where a_0 and $\Phi^{(0)}$ are the scale factor and the scalar field at $t = t_0$, $q < 2$ for inflation and

$$M = \frac{q}{2}\left(\frac{8\pi}{3m_p^2}\,\rho_{b\Phi}^{(0)}\right)^{1/2} \tag{5.3}$$

where $\rho_{b\Phi}^{(0)}$ is the homogeneous scalar field energy density at $t = t_0$. During "reheating" the JWKB (in m^{-1}) solutions are (eqns. (2.8), (2.10), (2.13), (2.19) and (2.20) of Ref. [13])

$$a_R(t) = a_1 \left[1 + \frac{3}{2} H_1 (t - t_1)\right]^{2/3}, \tag{5.4}$$

$$\Phi_{bR}(t) = \Phi^{(2)} + \frac{1}{[a_R(t)]^{3/2}}\left[C_+ \sin\{m(t-t_1)\} + C_- \cos\{m(t-t_1)\}\right], \tag{5.5}$$

where a_1 is the scale factor at $t = t_1$, $\Phi^{(2)}$ the scalar field at $t = t_2$, $C_\pm$ are constants and

$$H_1{}^2 = \frac{m^2}{12}\left(C_+^2 + C_-^2\right)\frac{1}{(a_1)^3}. \tag{5.6}$$

During kinetic energy dominance the solutions for the scale factor and the scalar field are given by eqns. (3.7) and (3.8) above.

Using the joining condition (4.10) to determine the constants of integration in the post-transition scale factors at the inflation-"reheating" transition (at $t = t_1$) and the "reheating" to kinetic energy dominance transition (at $t = t_2$) we find [from eqns. (3.7), (5.1) and (5.4)]

$$a_1 = a_0\left[1 + M(t_1 - t_0)\right]^{2/q}, \tag{5.7}$$

$$a_2 = a_1\left[1 + \frac{3}{2}H_1(t_2 - t_1)\right]^{2/3}. \tag{5.8}$$

Defining the present value of the scale factor

$$a_{\rm now} \equiv a_K(t_{\rm now}) = a_2\left[1 + \frac{\sqrt{3}}{2}\frac{C^{(K)}}{(a_2)^3}(t_{\rm now} - t_2)\right]^{1/3} \tag{5.9}$$

[$C^{(K)}$ is an integration constant, eqn. (3.6)], the redshift during kinetic energy dominance is

$$1 + z = \frac{\left[1 + \frac{\sqrt{3}}{2}\frac{C^{(K)}}{(a_2)^3}(t_{\rm now} - t_2)\right]^{1/3}}{\left[1 + \frac{\sqrt{3}}{2}\frac{C^{(K)}}{(a_2)^3}(t - t_2)\right]^{1/3}}, \tag{5.10}$$

that during "reheating" is given by

$$1 + z = \frac{\left[1 + \frac{3}{2}H_1(t_2 - t_1)\right]^{2/3}}{\left[1 + \frac{3}{2}H_1(t - t_1)\right]^{2/3}}\left[1 + \frac{\sqrt{3}}{2}\frac{C^{(K)}}{(a_2)^3}(t_{\rm now} - t_2)\right]^{1/3}, \tag{5.11}$$

while that during inflation is

$$1 + z = \frac{\left[1 + M(t_1 - t_0)\right]^{2/q}}{\left[1 + M(t - t_0)\right]^{2/q}}\left[1 + \frac{3}{2}H_1(t_2 - t_1)\right]^{2/3}\left[1 + \frac{\sqrt{3}}{2}\frac{C^{(K)}}{(a_2)^3}(t_{\rm now} - t_2)\right]^{1/3}. \tag{5.12}$$

From eqn. (5.2), the scalar field at the inflation-"reheating" transition (at $t = t_1$) is

$$\Phi^{(1)} = \Phi^{(0)} + \sqrt{\frac{8}{q}} \ln\left[1 + M(t_1 - t_0)\right]. \tag{5.13}$$

Using the joining condition (4.9) to determine a constant of integration in the posttransition homogeneous scalar field at each transition we find [from eqns. (3.8), (5.2), (5.5) and (5.13)]

$$\frac{C_-}{(a_1)^{3/2}} = \Phi^{(1)} - \Phi^{(2)}, \tag{5.14}$$

$$C_+ \sin\{m(t_2 - t_1)\} + C_- \cos\{m(t_2 - t_1)\} = 0. \tag{5.15}$$

Requiring that the two parts of the potential line up at the inflation-"reheating" transition, we have [from eqns. (2.4), (2.5), (5.1) and (5.2)]

$$m\left(\Phi^{(1)} - \Phi^{(2)}\right) = \sqrt{2(6-q)}\, H_I(t_1) \tag{5.16}$$

where $H_I(t)$ is the inflation epoch Hubble parameter. It may be verified that the parts of the potential match at the second transition [furthermore, since the "reheating" potential, eqn. (2.5), depends quadratically on the scalar field, V' does not jump at this transition]. For the assumption of Ref. [13], that during "reheating" $m \gg H$, to be valid, we must require $m \gg H_I(t_1)$ [this is sufficient since the Hubble parameter decreases with time] which, from eqn. (5.16), translates to the constraint

$$\Phi^{(1)} - \Phi^{(2)} \ll \sqrt{2(6-q)}, \tag{5.17}$$

that is, "reheating" must take place in a fairly small region of field space. Combining eqns. (5.14) and (5.16) we have

$$\frac{C_-}{(a_1)^{3/2}} = \sqrt{2(6-q)}\, \frac{H_1}{m} \tag{5.18}$$

where we have used H_1 to denote $H_I(t_1)$.

The joining conditions (4.10) and (4.22) result in the Hubble parameter not jumping at the transitions; they imply

$$H_1 = \frac{2}{q} \frac{M}{\left[1 + M(t_1 - t_0)\right]}, \tag{5.19}$$

$$\frac{C^{(K)}}{(a_2)^3} = 2\sqrt{3}\,\frac{H_1}{\left[1+\frac{3}{2}H_1(t_2-t_1)\right]}. \tag{5.20}$$

These relations ensure that the background energy density does not jump at the transitions (where we work to lowest order in m^{-1} during "reheating").

Matching the time derivative of the homogeneous scalar field at the inflation-"reheating" transition [eqn. (4.25)] and working to lowest order in m^{-1} we find

$$\frac{C_+}{(a_1)^{3/2}} = \sqrt{2q}\,\frac{H_1}{m}. \tag{5.21}$$

Eqns. (5.18) and (5.21) are consistent with eqn. (5.6); they also ensure that the background pressure does not jump at the inflation-"reheating" transition (to lowest order in m^{-1}). (We note that, as was assumed in Ref. [13], $C_\pm \propto m^{-1}$.) Combining these equations with eqn. (5.15) we find

$$\sin\{m(t_2-t_1)\} = -\left(\frac{6-q}{6}\right)^{1/2}, \tag{5.22}$$

$$\cos\{m(t_2-t_1)\} = \left(\frac{q}{6}\right)^{1/2}, \tag{5.23}$$

where we have assumed $t_2 \le t_1 + \pi/(2m)$ [i.e., that "reheating" is sufficiently rapid]. These equations along with eqns. (5.18) and (5.21) ensure that the background pressure does not jump at the transition from "reheating" to kinetic energy dominance (to lowest order in m^{-1}). These relations [with eqn. (5.20)] also ensure that the time derivative of the spatially homogeneous part of the scalar field does not jump at this transition.

We now turn to determining the number of free parameters, of large-scale cosmological interest, in this model. Since the model is spatially flat $\Omega_{\rm now} = 1$; we also assume that either $H_{\rm now}$ [$\equiv H_K(t_{\rm now})$] or $\rho_{\rm now}$ [$\equiv \rho_{b\Phi(K)}(t_{\rm now})$] has a given numerical value. [The density parameter Ω is the ratio of the mean mass density to the Einstein-de Sitter value, $\Omega(t) = 8\pi G\rho/(3H^2)$; $\Omega_{\rm now}$ denotes the present value of the density parameter.] It is straightforward to establish that in the three epochs of the model the Hubble parameter is given by

$$H_K(t) = H_{\rm now}(1+z)^3, \tag{5.24}$$

$$H_R(t) = H_2 \left(\frac{1+z}{1+z_2} \right)^{3/2} , \tag{5.25}$$

$$H_I(t) = H_1 \left(\frac{1+z}{1+z_1} \right)^{q/2} . \tag{5.26}$$

Given $H_{\rm now}$ and z_2, the redshift of the "reheating" to kinetic energy domination transition, eqn. (5.24) determines the Hubble parameter, H_2, at z_2; in turn, given z_1 (the inflation-"reheating" redshift) eqn. (5.25) fixes the Hubble parameter at this redshift, H_1. To get the Hubble parameter during inflation from eqn. (5.26) one requires the value of q. It may be verified that these three parameters (z_1, z_2 and q) determine the homogeneous scalar field for all redshift provided one has numerical values for $\Phi_{\rm now}$ [$\equiv \Phi_{bK}(t_{\rm now})$] and m [or $\Phi^{(1)}$].

We shall now argue that these three parameters, z_1, z_2 and q [subject to the constraint of eqn. (5.17) or equivalently $m \gg H_1$, which may be reexpressed, using eqns. (5.24) and (5.25), as the constraint $z_1 \times z_2 \ll (m/H_{\rm now})^{2/3}$; roughly, if we take $H_{\rm now} = 100\, h\, {\rm km\, s^{-1}\, Mpc^{-1}}$, this becomes $z_1 \times z_2 \ll 10^{38} [m(10^{15}\ {\rm GeV})/h]^{2/3}$, a not too unreasonably small a value], are the only free parameters of large-scale interest. It was shown in Ref. [8] that the solution of the exponential potential inflation model is a stable (time-dependent) fixed point — in particular, the energy density irregularity power spectrum cannot depend on the initial value of the spatially homogeneous part of the scalar field (this was confirmed by the calculations of Refs. [1,2]). Since expressions for the variables of interest can only depend on the background scalar field through differences in the field (and since they do not depend on the initial value of the background scalar field) there are only three free parameters of cosmological interest (this will be quantitatively verified below). This is not true in the quartic potential scalar field inflation model (as was shown explicitly in Ref. [3]), since in this case the solution is not a stable fixed point. If we had used, in a calculation similar to that of this paper, the quartic model to describe the initial inflation epoch we would have found that the fractional energy density power spectrum depended on four free parameters (instead of three).

5.2 Joining the Spatial Irregularities: the Inflation-"Reheating" Transition

The general procedure is similar to that used in Sec. 4.1 of Ref. [1], with one caveat — in the analysis of the "reheating" model, [13], we made use of a JWKB approximation which effectively ignored spatial gradients in the scalar field, this means that the scalar field results of Ref. [13] are only strictly valid for super-Hubble-scale modes. The constants in the inflation epoch solutions for ϕ_I and $h_{22}^{(I)} + h_{33}^{(I)}$ have been determined from the quantum mechanical initial conditions of Ref. [4] in Sec. 3.3 of Ref. [1]. In this section we determine the constants in the "reheating" epoch expressions for irregularities.

In the "reheating" epoch the scalar field perturbation is (eqns. (2.10), (2.28), (2.34), (2.39) and (2.40) [where we have set $c_3 = 0$] of Ref. [13] and eqns. (5.18) and (5.21) above)

$$\phi_R(t) = \phi_+(t) \sin\{m(t - t_1)\} + \phi_-(t) \cos\{m(t - t_1)\}, \tag{5.27}$$

where the functions $\phi_\pm$ are

$$\phi_+(x) = c_1^{(R)} e^{-x/3} + c_2^{(R)} e^{-x} + c_4^{(R)} e^{-2x}, \tag{5.28}$$

$$\phi_-(x) = \left(\frac{6-q}{q}\right)^{1/2} \left[c_1^{(R)} e^{-x/3} - c_2^{(R)} \left(\frac{q}{6-q}\right) e^{-x} + c_4^{(R)} e^{-2x}\right], \tag{5.29}$$

$c_1^{(R)}$, $c_2^{(R)}$ and $c_4^{(R)}$ are constants of integration [in these equations the terms proportional to $c_2^{(R)}$ and (part of) those proportional to $c_4^{(R)}$ are gauge-dependent] and

$$e^x = t - t_1 + \frac{2}{3}\frac{1}{H_1}. \tag{5.30}$$

Using the joining conditions for ϕ, $\dot{\phi}$ and $(h_{22} + h_{33})$, eqns. (4.72), (4.74) and (4.81) [it proves convenient to transpose the second term in the square parentheses on the right hand side of eqn. (4.74) to the left hand side and to use eqns. (4.72) and (4.25) to replace ϕ_I by ϕ_R and $\dot{\Phi}_{bI}$ by $\dot{\Phi}_{bR}$], the results of Sec. 3.3 and eqns. (2.5), (3.5), (3.10), (3.19) [where we set $c_7 = 0 = c_8$], (3.40) and (3.43) of Ref. [1], eqns. (2.48), (2.62) and (2.65) of Ref. [13], and the results of Sec. 5.1 above, we find that the constants of integration are given by

$$c_1^{(R)} = \frac{3}{10}\left(\frac{\lambda_R}{m}\right)^2 \Bigg[-\frac{3}{q}\left(\frac{6-q}{q}\right)^{1/2} c_2\, e^{-2\bar{x}_1} \tag{5.31}$$
$$+ m\,\frac{q}{2-q}\, c_+\, \lambda^{-1}\, e^{-(6-q)\bar{x}_1/(2q)}\, H^{(1)}_{\nu+1}\left(\lambda\, e^{-(2-q)\bar{x}_1/q}\right) + \cdots \Bigg],$$

$$c_2^{(R)} = -\frac{q}{3}\left(\frac{6-q}{q}\right)^{1/2} \left[c_2 + \frac{q}{2-q}\, c_+\, \lambda^{-3/2}\, e^{-(2-q)\bar{x}_1/q}\, G_1(\bar{x}_1) + \cdots \right], \tag{5.32}$$

$$c_4^{(R)} = -\frac{q}{3}\frac{1}{m}\Bigg[c_2 + \frac{q}{2-q}\, c_+\, \lambda^{-3/2}\, e^{-(2-q)\bar{x}_1/q} \Bigg\{ G_1(\bar{x}_1) \tag{5.33}$$
$$- \frac{q}{18}(2-q)\,\lambda^{3/2}\, e^{-3(2-q)\bar{x}_1/(2q)}\, H^{(1)}_{\nu}\left(\lambda\, e^{-(2-q)\bar{x}_1/q}\right)$$
$$+ \frac{(2-q)^2}{30}\,\lambda^{5/2}\, e^{-5(2-q)\bar{x}_1/(2q)}\, H^{(1)}_{\nu+1}\left(\lambda\, e^{-(2-q)\bar{x}_1/q}\right) \Bigg\} + \cdots \Bigg].$$

In these expressions we have kept the leading (in m^{-1}) contribution to both the c_2 and c_+ terms and the ellipses denote higher order terms (we note that, consistent with the assumption of Ref. [13], the leading terms in these equations are at most of order m^0). The various symbols in these equations are defined by

$$e^{x_1} = \frac{2}{3}\frac{1}{H_1}, \tag{5.34}$$

$$e^{\bar{x}_1} = \frac{3}{q} e^{x_1}, \tag{5.35}$$

$$\lambda_R = \frac{k}{a_1}\left(\frac{2}{3H_1}\right)^{2/3}, \tag{5.36}$$

c_+ is the inflation epoch (adiabatic solution) constant of integration —

$$c_+ = \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{k}{2}\left(\frac{q\pi}{2-q}\right)^{1/2} \left(a_0 M^{2/q}\right)^{-5/2} e^{i(\nu-1/2)\pi/2} \tag{5.37}$$

(the inflation epoch gauge-dependent time translation invariance solution constant of integration, c_2, is left undetermined by the quantum mechanical initial conditions, [1]), and

$$G_1(x) = H^{(1)}_{\nu-1}\left(\lambda\, e^{-(2-q)x/q}\right) S_{3/2,\nu}\left(\lambda\, e^{-(2-q)x/q}\right) \tag{5.38}$$
$$- \left(\nu + \frac{1}{2}\right) H^{(1)}_{\nu}\left(\lambda\, e^{-(2-q)x/q}\right) S_{1/2,\nu-1}\left(\lambda\, e^{-(2-q)x/q}\right),$$

where $H^{(1)}_\mu$ is a Hankel function, $S_{\mu,\rho}$ is a Lommel function, and we have defined

$$\nu = \frac{1}{2}\left(\frac{2+q}{2-q}\right), \tag{5.39}$$

$$\lambda = \frac{q}{2-q}\,\frac{k}{a_0 M^{2/q}}. \tag{5.40}$$

To simplify the expression for $c_4^{(R)}$, eqn. (5.33), we have made use of the relation

$$\lambda_R\, e^{x_1/3} = \frac{2-q}{3}\,\lambda\, e^{-(2-q)\bar{x}_1/q}. \tag{5.41}$$

It is pleasing to note that the c_2 contribution (from the inflation epoch gauge-dependent time translation solution) to the expression for $c_1^{(R)}$ [the constant of integration for the "reheating" epoch adiabatic solution, eqn. (5.31)] is down by a factor of m^{-1} relative to the other term in the square parentheses of eqn. (5.31); this provides a useful partial check of our formalism and manipulations. (A discussion of the expression for $c_4^{(R)}$ is given below.)

The constants in the "reheating" solutions may also be determined by using the equivalent spacetime-dependent "speed of sound" fluid joining conditions for δ_Φ, v^1_Φ and $(h_{22}+h_{33})$, eqn. (4.81), the $i=1$ part of eqn. (4.87) and eqn. (4.89) [since $\rho_{b\Phi}$ and $p_{b\Phi}$ do not jump at the transition we actually join $\rho_{b\Phi}\delta_\Phi$ and $(\rho_{b\Phi}+p_{b\Phi})v^1_\Phi$]. From the results of Refs. [1,13] we find that the energy density perturbation in the "reheating" epoch is

$$\begin{aligned}
\rho_{b\Phi(R)}(x)\,\delta_{\Phi(R)}(x) = \frac{m_p^2}{4\pi}\Bigg[&\sqrt{\frac{2}{q}}\,m\left(c_1^{(R)}\,e^{-4x/3} + c_4^{(R)}\,e^{-3x}\right) \\
&-\sqrt{\frac{6-q}{2}}\left(\frac{4}{9}\,c_1^{(R)}\,e^{-7x/3} + \frac{2}{3}\left(\frac{3-q}{6-q}\right)c_2^{(R)}\,e^{-3x} + c_4^{(R)}\,e^{-4x}\right) \\
&\times\left(\cos^2\{m(t-t_1)\} - \sin^2\{m(t-t_1)\}\right) \\
&+\sqrt{\frac{2}{q}}\left(\frac{4}{9}(3-q)\,c_1^{(R)}\,e^{-7x/3} - \frac{2}{3}\,q\,c_2^{(R)}\,e^{-3x} + (3-q)\,c_4^{(R)}e^{-4x}\right) \\
&\times\cos\{m(t-t_1)\}\sin\{m(t-t_1)\} + \cdots\Bigg],
\end{aligned} \tag{5.42}$$

where the constants of integration are given by eqns. (5.31) – (5.33); this result provides an alternate check on our algebraic manipulations.

One may also use the joining conditions for gauge-invariant variables, eqns. (4.92) and (4.93), to determine some of these constants of integration (it proves to be convenient to join $\rho_{b\Phi}\Delta_\Phi$ and $\rho_{b\Phi}A_\Phi$ at the transition). We find, in the "reheating" epoch, that the gauge-invariant measure of energy density perturbations is

$$\rho_{b\Phi(R)}(x)\,\Delta_{\Phi(R)}(x) = \frac{m_p^2}{8\pi}\Bigg[2\sqrt{\frac{2}{q}}\;m\left(c_1^{(R)}\,e^{-4x/3} + c_4^{(R)}\,e^{-3x}\right) - \sqrt{\frac{2}{6-q}}\;c_2^{(R)}\left(\lambda_R^2\,e^{-7x/3} + 2\,e^{-3x}\right) + \cdots\Bigg], \tag{5.43}$$

while the second gauge-invariant variable is given by

$$\begin{aligned}\rho_{b\Phi(R)}(x)\,A_{\Phi(R)}(x) = \frac{m_p^2}{8\pi}\Bigg[&2\sqrt{\frac{2}{q}}\;m\left\{c_1^{(R)}\left(e^{-4x/3} + \frac{10}{3}\frac{e^{-2x}}{\lambda_R^2}\right) + c_4^{(R)}\,e^{-3x}\right\}\\ &-\frac{20}{9}\sqrt{\frac{2}{q}}\,\frac{m}{\lambda_R^2}c_1^{(R)}\,e^{-2x}\Big\{(3-q)\left(\cos^2\{m(t-t_1)\} - \sin^2\{m(t-t_1)\}\right)\\ &\qquad + 2\sqrt{q(6-q)}\cos\{m(t-t_1)\}\sin\{m(t-t_1)\}\Big\}\\ &-2\sqrt{\frac{2}{6-q}}\;c_2^{(R)}\,e^{-3x}\\ &-\frac{2}{3}\left\{\frac{4}{3}c_1^{(R)}\left(e^{-7x/3} + 10\frac{e^{-3x}}{\lambda_R^2}\right) + 3c_4^{(R)}\,e^{-4x}\right\}\\ &\times\Bigg\{\sqrt{\frac{6-q}{2}}\left(\cos^2\{m(t-t_1)\} - \sin^2\{m(t-t_1)\}\right)\\ &\qquad -(3-q)\sqrt{\frac{2}{q}}\cos\{m(t-t_1)\}\sin\{m(t-t_1)\}\Bigg\} + \cdots\Bigg].\end{aligned} \tag{5.44}$$

Working to lowest order in m^{-1} we find that the constants of integration are

$$c_1^{(R)} = \frac{3}{10}\left(\frac{\lambda_R}{m}\right)^2\left[m\,\frac{q}{2-q}\,c_+\,\lambda^{-1}\,e^{-(6-q)\bar{x}_1/(2q)}\,H_{\nu+1}^{(1)}\left(\lambda\,e^{-(2-q)\bar{x}_1/q}\right) + \cdots\right] \tag{5.45}$$

[which coincides with the (dominant) second term in the square parentheses in eqn. (5.31)] and

$$\begin{aligned}c_4^{(R)} = -\frac{q}{3}\frac{1}{m}\Bigg[&\frac{q}{2-q}\,c_+\,\lambda^{-3/2}\,e^{-(2-q)\bar{x}_1/q}\\ &\times\Bigg\{-\frac{q}{18}\,(2-q)\,\lambda^{3/2}\,e^{-3(2-q)\bar{x}_1/(2q)}\,H_\nu^{(1)}\left(\lambda\,e^{-(2-q)\bar{x}_1/q}\right)\end{aligned} \tag{5.46}$$

$$+\frac{(2-q)^2}{30}\,\lambda^{5/2}\,e^{-5(2-q)\bar{x}_1/(2q)}\,H^{(1)}_{\nu+1}\left(\lambda\,e^{-(2-q)\bar{x}_1/q}\right)\Big\}+\cdots\Big]$$

which coincides with the last two terms in eqn. (5.33).

It is pleasing to note (as discussed in Sec. 2.5 of Ref. [13]) that, in both eqn. (5.43) and eqn. (5.44), part of the order m^0 part of the solution proportional to $c_2^{(R)}$ (the term proportional to e^{-3x}) may be added to the order $m\,c_4^{(R)}$ solution to yield a gauge-invariant constant of integration $c_4^{(R)} - \sqrt{q/(6-q)}\,\{c_2^{(R)}/m\}$; using eqns. (5.32) and (5.33) we find

$$\begin{aligned} c_4^{(R)} - \sqrt{\frac{q}{6-q}}\,\frac{c_2^{(R)}}{m} = -\frac{q}{3}\frac{1}{m}\Bigg[&\frac{q}{2-q}\,c_+\,\lambda^{-3/2}\,e^{-(2-q)\bar{x}_1/q} \\ &\times\Bigg\{-\frac{q}{18}\,(2-q)\,\lambda^{3/2}\,e^{-3(2-q)\bar{x}_1/(2q)}\,H^{(1)}_{\nu}\left(\lambda\,e^{-(2-q)\bar{x}_1/q}\right) \\ &+\frac{(2-q)^2}{30}\,\lambda^{5/2}\,e^{-5(2-q)\bar{x}_1/(2q)}\,H^{(1)}_{\nu+1}\left(\lambda\,e^{-(2-q)\bar{x}_1/q}\right)\Bigg\}+\cdots\Bigg], \end{aligned} \tag{5.47}$$

the right hand side of which is identical to that of eqn. (5.46) — this result provides yet another check on our formalism and algebraic manipulations. (That the expression for $c_4^{(R)}$ given in eqn. (5.33) depends on the inflation epoch gauge-dependent constant of integration c_2 while the expression given in eqn. (5.46) does not, is a reflection both of the standard peculiarity of the gauge transformation property of the decaying solution in a pressureless fluid as well as the order (in m^{-1}) to which we have consistently worked in Ref. [13].)

The trace of the metric perturbation is not of further interest so we ignore the joining condition (4.75). From the joining condition (4.77) we have

$$h_{I1}^{(R)}(x) = c_{I,1}^{(R)} \tag{5.48}$$

where the "reheating" epoch constant of integration is related to that during inflation by

$$c_{I,1}^{(R)} = c_{I,1}; \tag{5.49}$$

it is obvious that the consistency condition (4.78) is satisfied. We note that $c_{I,1}^{(R)}$ and $c_{I,1}$ correspond to gauge-dependent solutions and that the inflation epoch quantum mechanical initial conditions do not fix $c_{I,1}$. The constants in the inflation epoch gravitational wave solutions have been determined from quantum mechanical initial conditions in Ref. [1]; the joining conditions (4.79) and (4.80) determine the gravitational wave constants during "reheating". We find,

$$h_i^{(R)}(x) = e^{-x/2}\left[c_{i1}^{(R)} H_{3/2}^{(1)}\left(3\lambda_R\, e^{x/3}\right) + c_{i2}^{(R)} H_{3/2}^{(2)}\left(3\lambda_R e^{x/3}\right)\right] \tag{5.50}$$

(here $i = +$ or $\times$) where the constants of integration are given by

$$\begin{aligned} c_{i1}^{(R)} =& -\frac{i\pi}{4}\,(2-q)\sqrt{\frac{q}{3}}\,c_{i1}\,\lambda\, e^{-(5-2q)\bar{x}_1/q} \\ &\times\left[H_{5/2}^{(2)}\left(3\lambda_R\, e^{x_1/3}\right)\, H_{\nu+1}^{(1)}\left(\lambda\, e^{-(2-q)\bar{x}_1/q}\right) - H_{3/2}^{(2)}\left(3\lambda_R e^{x_1/3}\right)\, H_{\nu}^{(1)}\left(\lambda\, e^{-(2-q)\bar{x}_1/q}\right)\right], \end{aligned} \tag{5.51}$$

$$\begin{aligned} c_{i2}^{(R)} =& \frac{i\pi}{4}\,(2-q)\sqrt{\frac{q}{3}}\,c_{i1}\,\lambda\, e^{-(5-2q)\bar{x}_1/q} \\ &\times\left[H_{5/2}^{(1)}\left(3\lambda_R\, e^{x_1/3}\right)\, H_{\nu+1}^{(1)}\left(\lambda\, e^{-(2-q)\bar{x}_1/q}\right) - H_{3/2}^{(1)}\left(3\lambda_R e^{x_1/3}\right)\, H_{\nu}^{(1)}\left(\lambda\, e^{-(2-q)\bar{x}_1/q}\right)\right], \end{aligned} \tag{5.52}$$

and we have defined

$$c_{i1} = \left(\frac{16\pi}{m_p^2}\right)^{1/2}\left(\frac{q\pi}{2-q}\right)^{1/2}\left(a_0 M^{2/q}\right)^{-3/2} e^{i(\nu+3/2)\pi/2}. \tag{5.53}$$

5.3 Joining the Spatial Irregularities: the "Reheating" to Kinetic Energy Dominance Transition

Using the joining conditions for the scalar field perturbation, for its time derivative and for the metric perturbation combination $(h_{22} + h_{33})$, eqns. (4.72), (4.74) and (4.81) above (with the replacements $R \to K$, $I \to R$ and $t_1 \to t_2$), we find that the constants of integration in the scalar field irregularity solution [eqns. (3.20) – (3.22)] are given by

$$\begin{aligned} c_2^{(K)} =& \frac{3}{16}\left(\frac{6}{q}\right)^{1/2} i\pi\,\frac{m}{\lambda_K^{1/2}}\, e^{2x_2/3} \\ &\times\left[c_1^{(R)}\, e^{2\bar{x}_2/3}\left\{G_1^K(x_2)\left(H_0^{(2)}\left(\lambda_K\, e^{2x_2/3}\right) - \frac{5}{2}\,\lambda_K^{-1}\, e^{-2x_2/3}\, H_1^{(2)}\left(\lambda_K\, e^{2x_2/3}\right)\right)\right.\right. \end{aligned} \tag{5.54}$$

$$
\begin{aligned}
&\qquad\qquad - G_2^K(x_2) \left(H_0^{(1)} \left(\lambda_K \, e^{2x_2/3} \right) - \frac{5}{2} \, \lambda_K^{-1} \, e^{-2x_2/3} \, H_1^{(1)} \left(\lambda_K \, e^{2x_2/3} \right) \right) \Bigg\} \\
&\quad + \frac{8}{3} \left(\frac{q}{6-q} \right)^{1/2} \frac{i}{\pi} \, c_2^{(R)} \, \frac{\lambda_K^{1/2}}{m} \, e^{-2x_2/3} \\
&\quad + \left\{ c_4^{(R)} - \sqrt{\frac{q}{6-q}} \, \frac{c_2^{(R)}}{m} \right\} e^{-\bar{x}_2} \\
&\quad \times \left\{ G_1^K(x_2) \, H_0^{(2)} \left(\lambda_K \, e^{2x_2/3} \right) - G_2^K(x_2) \, H_0^{(1)} \left(\lambda_K \, e^{2x_2/3} \right) \right\} + \cdots \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
c_+^{(K)} =& \frac{1}{8} \left(\frac{6}{q} \right)^{1/2} i\pi \, m \, \lambda_K \\
& \times \left[c_1^{(R)} \, e^{2\bar{x}_2/3} \left\{ H_0^{(2)} \left(\lambda_K \, e^{2x_2/3} \right) - \frac{5}{2} \, \lambda_K^{-1} \, e^{-2x_2/3} \, H_1^{(2)} \left(\lambda_K \, e^{2x_2/3} \right) \right\} \right. \\
& \quad \left. + \left\{ c_4^{(R)} - \sqrt{\frac{q}{6-q}} \, \frac{c_2^{(R)}}{m} \right\} e^{-\bar{x}_2} \, H_0^{(2)} \left(\lambda_K \, e^{2x_2/3} \right) + \cdots \right],
\end{aligned}
\tag{5.55}
$$

$$
\begin{aligned}
c_-^{(K)} =& -\frac{1}{8} \left(\frac{6}{q} \right)^{1/2} i\pi \, m \, \lambda_K \\
& \times \left[c_1^{(R)} \, e^{2\bar{x}_2/3} \left\{ H_0^{(1)} \left(\lambda_K \, e^{2x_2/3} \right) - \frac{5}{2} \, \lambda_K^{-1} \, e^{-2x_2/3} \, H_1^{(1)} \left(\lambda_K \, e^{2x_2/3} \right) \right\} \right. \\
& \quad \left. + \left\{ c_4^{(R)} - \sqrt{\frac{q}{6-q}} \, \frac{c_2^{(R)}}{m} \right\} e^{-\bar{x}_2} \, H_0^{(1)} \left(\lambda_K \, e^{2x_2/3} \right) + \cdots \right],
\end{aligned}
\tag{5.56}
$$

where we have retained the leading (in m^{-1}) $c_1^{(R)}$, $c_2^{(R)}$ and $c_4^{(R)}$ contributions and the ellipses denote higher order terms. In these expressions we have defined

$$
e^{x_2} = \frac{1}{3H_2} = \frac{1}{2} e^{\bar{x}_2},
\tag{5.57}
$$

G_1^K and G_2^K are given in eqns. (3.21) and (3.22) and $c_1^{(R)}$, $c_2^{(R)}$ and $c_4^{(R)}$ are defined by eqns. (5.31) – (5.33). To simplify the above expressions for $c_2^{(K)}$ and $c_\pm^{(K)}$ we have used

$$
\lambda_K \, e^{2x_2/3} = \frac{3}{4} \, \lambda_R \, e^{\bar{x}_2/3}.
\tag{5.58}
$$

We note that the contribution from the "reheating" epoch gauge-dependent time translation invariance solution (the terms proportional to $c_2^{(R)}$) in eqns. (5.55) and (5.56) are suppressed by a factor of m^{-1} relative to the other terms in these equations and that when $c_2^{(R)}$ does appear in eqns. (5.55) and (5.56) it appears in the gauge-invariant combination

$c_4^{(R)} - \sqrt{q/(6-q)}\,\{c_2^{(R)}/m\}$ which is independent of the inflation epoch gauge-dependent time translation solution constant c_2 [see eqn. (5.47)]. Eqns. (5.54) – (5.56) may also be derived by making use of the equivalent spacetime-dependent "speed of sound" fluid model joining conditions for δ_Φ, v^1_Φ and $(h_{22}+h_{33})$; this provides a check on our computations.

Expressions for the constants of integration which correspond to gauge-invariant solutions, $c_\pm^{(K)}$, may also be derived by joining the gauge-invariant variables Δ_Φ and A_Φ, eqns. (4.92) and (4.93). We find, to leading order in m^{-1},

$$\begin{aligned}
c_+^{(K)} =& \frac{1}{8}\left(\frac{6}{q}\right)^{1/2} i\pi\, m\,\lambda_K \\
&\times\left[c_1^{(R)}\, e^{2\bar{x}_2/3}\left\{H_0^{(2)}\left(\lambda_K\, e^{2x_2/3}\right) - \frac{5}{2}\lambda_K^{-1}\, e^{-2x_2/3}\, H_1^{(2)}\left(\lambda_K\, e^{2x_2/3}\right)\right\}\right. \\
&\left.+\, c_4^{(R)}\, e^{-\bar{x}_2}\, H_0^{(2)}\left(\lambda_K\, e^{2x_2/3}\right) + \cdots\right],
\end{aligned} \tag{5.59}$$

$$\begin{aligned}
c_-^{(K)} =& -\frac{1}{8}\left(\frac{6}{q}\right)^{1/2} i\pi\, m\,\lambda_K \\
&\times\left[c_1^{(R)}\, e^{2\bar{x}_2/3}\left\{H_0^{(1)}\left(\lambda_K\, e^{2x_2/3}\right) - \frac{5}{2}\lambda_K^{-1}\, e^{-2x_2/3}\, H_1^{(1)}\left(\lambda_K\, e^{2x_2/3}\right)\right\}\right. \\
&\left.+\, c_4^{(R)}\, e^{-\bar{x}_2}\, H_0^{(1)}\left(\lambda_K\, e^{2x_2/3}\right) + \cdots\right];
\end{aligned} \tag{5.60}$$

the expression for $c_4^{(R)}$ that must be used in these equations is that given in eqn. (5.46). Alternatively, these equations coincide with eqns. (5.55) and (5.56) if one makes the replacement $c_4^{(R)} \to c_4^{(R)} - \sqrt{q/(6-q)}\,\{c_2^{(R)}/m\}$ [see the discussion around eqn. (5.47)].

The metric perturbation trace is not of further interest so we ignore the joining condition (4.75). From the joining condition (4.77) we find that the constant in eqn. (3.33) is

$$c_{I,1}^{(K)} = c_{I,1}^{(R)}. \tag{5.61}$$

It is straightforward to establish that the consistency condition (4.78) is satisfied. The joining conditions (4.79) and (4.80) determine the gravitational wave constants in the kinetic energy dominated epoch. We find that the constants in eqn. (3.34) are given by

$$
\begin{aligned}
c_{i1}^{(K)} =& -\frac{3}{16}\, i\pi\, \lambda_R\, e^{-\bar{x}_2/6} \qquad (5.62)\\
&\times \Bigg[H_1^{(2)}\left(\lambda_K\, e^{2x_2/3}\right) \left\{ c_{i1}^{(R)} H_{3/2}^{(1)}\left(3\lambda_R\, e^{\bar{x}_2/3}\right) + c_{i2}^{(R)} H_{3/2}^{(2)}\left(3\lambda_R\, e^{\bar{x}_2/3}\right)\right\}\\
&\quad - H_0^{(2)}\left(\lambda_K\, e^{2x_2/3}\right) \left\{ c_{i1}^{(R)} H_{5/2}^{(1)}\left(3\lambda_R\, e^{\bar{x}_2/3}\right) + c_{i2}^{(R)} H_{5/2}^{(2)}\left(3\lambda_R e^{\bar{x}_2/3}\right)\right\} \Bigg],\\
c_{i2}^{(K)} =& \frac{3}{16}\, i\pi\, \lambda_R\, e^{-\bar{x}_2/6} \qquad (5.63)\\
&\times \Bigg[H_1^{(1)}\left(\lambda_K\, e^{2x_2/3}\right) \left\{ c_{i1}^{(R)} H_{3/2}^{(1)}\left(3\lambda_R\, e^{\bar{x}_2/3}\right) + c_{i2}^{(R)} H_{3/2}^{(2)}\left(3\lambda_R\, e^{\bar{x}_2/3}\right)\right\}\\
&\quad - H_0^{(1)}\left(\lambda_K\, e^{2x_2/3}\right) \left\{ c_{i1}^{(R)} H_{5/2}^{(1)}\left(3\lambda_R\, e^{\bar{x}_2/3}\right) + c_{i2}^{(R)} H_{5/2}^{(2)}\left(3\lambda_R e^{\bar{x}_2/3}\right)\right\} \Bigg],
\end{aligned}
$$

where i is either $+$ or $\times$ and $c_{i1}^{(R)}$ and $c_{i2}^{(R)}$ are given in eqns. (5.51) and (5.52).

6. Intermediate and Large Times

In this section we evaluate the power spectrum of fractional energy density irregularities and the gravitational wave perturbation two-point function in the "reheating" and kinetic energy dominated epochs (and examine these quantities in the de Sitter inflation limit).

6.1 Large-Scale Approximation

Perturbations on scales larger than the Hubble scale during the inflation-"reheating" transition and the "reheating" to kinetic energy dominance transition are not affected by small-scale microphysical processes (which we have ignored) and are hence characteristic of the inflation epoch of the model. These perturbations are the leading terms of the solutions we have derived in the limit of small $k/(a_1\, H_1)$ and $k/(a_2\, H_2)$.

Working to leading order in m^{-1}, we find that the large-scale terms in the constants which characterize matter irregularities during "reheating", eqns. (5.31) – (5.33), are

$$
c_1^{(R)} = \frac{1}{m}\left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{\sqrt{2\pi}}{15}\left(\frac{3}{2}\right)^{2/3} (2-q)^{\nu+1/2}\, \frac{\csc(\nu\pi)}{\Gamma(-\nu)}\, e^{i(\nu+1/2)\pi/2}
$$

$$\times\, k^{-3/2} \left(\frac{k}{a_1\, H_1}\right)^{5/2-\nu} (H_1)^{5/3} + \cdots\,, \tag{6.1}$$

$$\begin{aligned} c_2^{(R)} =& -\frac{q}{3}\left(\frac{6-q}{q}\right)^{1/2} c_2 \\ & -\left(\frac{16\pi}{m_p^2}\right)^{1/2} i\left[\frac{2}{\pi}\left(\frac{6-q}{q}\right)\right]^{1/2}\left(\frac{2-q}{3}\right)\Gamma\!\left(\frac{5}{4}-\frac{\nu}{2}\right)\Gamma\!\left(\frac{5}{4}+\frac{\nu}{2}\right)k^{-3/2} \\ & -\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{2}{3}\left[2\pi\left(\frac{6-q}{q}\right)\right]^{1/2}\frac{(2-q)^{\nu-1/2}}{(4-3q)}\frac{\csc(\nu\pi)}{\Gamma(1-\nu)}\,e^{i(\nu+1/2)\pi/2} \\ & \times\, k^{-3/2}\left(\frac{k}{a_1\, H_1}\right)^{5/2-\nu} + \cdots\,, \end{aligned} \tag{6.2}$$

$$\begin{aligned} c_4^{(R)} =& -\frac{1}{m}\,\frac{q}{3}\,c_2 - \frac{1}{m}\left(\frac{16\pi}{m_p^2}\right)^{1/2} i\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{2-q}{3}\right)\Gamma\!\left(\frac{5}{4}-\frac{\nu}{2}\right)\Gamma\!\left(\frac{5}{4}+\frac{\nu}{2}\right)k^{-3/2} \\ & -\frac{1}{m}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{4}{135}\sqrt{2\pi}\,\frac{(2-q)^{\nu+1/2}}{(4-3q)}\,\frac{(3q^2-13q-33)}{(2+q)}\,\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\,e^{i(\nu+1/2)\pi/2} \\ & \times\, k^{-3/2}\left(\frac{k}{a_1\, H_1}\right)^{5/2-\nu} + \cdots\,. \end{aligned} \tag{6.3}$$

In these equations a_1 and H_1 are the scale factor and the Hubble parameter at the inflation-"reheating" transition, c_2 is the inflation epoch time translation solution constant, ν is defined in eqn. (5.39) and the ellipses now also denote terms that are down by positive powers of spatial momentum. The expansion of the combination of eqn. (5.47) is

$$\begin{aligned} c_4^{(R)} - \sqrt{\frac{q}{6-q}}\,\frac{c_2^{(R)}}{m} =& -\frac{1}{m}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{4}{135}\sqrt{2\pi}\left(\frac{3-q}{2+q}\right)(2-q)^{\nu+1/2}\,\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\,e^{i(\nu+1/2)\pi/2} \\ & \times\, k^{-3/2}\left(\frac{k}{a_1\, H_1}\right)^{5/2-\nu} + \cdots\,. \end{aligned} \tag{6.4}$$

During kinetic energy dominance, the large-scale part of the constants of integration in the solutions for matter inhomogeneities, eqns. (5.54) – (5.56), are described by

$$\begin{aligned} c_2^{(K)} =& \left(\frac{q}{6}\right)^{1/2} c_2 + \left(\frac{16\pi}{m_p^2}\right)^{1/2} i\left(\frac{3}{q\pi}\right)^{1/2}\left(\frac{2-q}{3}\right)\Gamma\!\left(\frac{5}{4}-\frac{\nu}{2}\right)\Gamma\!\left(\frac{5}{4}+\frac{\nu}{2}\right)k^{-3/2} \\ & +\left(\frac{16\pi}{m_p^2}\right)^{1/2}\left(\frac{6}{q\pi}\right)^{1/2}\Gamma^2\!\left(\frac{3}{4}\right)(2-q)^{\nu+1/2}\,\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\,e^{i(\nu+1/2)\pi/2} \\ & \times\, k^{-3/2}\left(\frac{k}{a_1\, H_1}\right)^{5/2-\nu}\left(\frac{k}{a_2\, H_2}\right)^{-7/2}\left(\frac{H_1}{H_2}\right)^{5/3} + \cdots\,, \end{aligned} \tag{6.5}$$

$$c_{\pm}^{(K)} = \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{(3)^{2/3}}{6} \left(\frac{3\pi}{q}\right)^{1/2} (2-q)^{\nu+1/2} \frac{\csc(\nu\pi)}{\Gamma(-\nu)}\, e^{i(\nu+1/2)\pi/2}$$
$$\times\, k^{-3/2} \left(\frac{k}{a_1\, H_1}\right)^{5/2-\nu} \left(\frac{k}{a_2\, H_2}\right)^{-1} (H_1)^{5/3} + \cdots, \tag{6.6}$$

where a_2 and H_2 are the scale factor and the Hubble parameter at the second transition.

The large-scale asymptotic form of the constants of integration which characterize gravitational wave perturbations during "reheating", eqns. (5.51) and (5.52), is

$$c_{i\left({1 \atop 2}\right)}^{(R)} = \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{\pi}{4} \sqrt{3}\, (2-q)^{\nu+1/2} \frac{\csc(\nu\pi)}{\Gamma(-\nu)}\, e^{i(\nu-3/2)\pi/2}$$
$$\times\, k^{-3/2} \left(\frac{k}{a_1\, H_1}\right)^{-1-\nu} (H_1)^{1/2} + \cdots, \tag{6.7}$$

while those that characterize these large-scale perturbations during kinetic energy dominance, eqns. (5.62) and (5.63), become

$$c_{i\left({1 \atop 2}\right)}^{(K)} = -\left(\frac{16\pi}{m_p^2}\right)^{1/2} \left(\frac{\pi}{2}\right)^{1/2} (2-q)^{\nu+1/2} \frac{\csc(\nu\pi)}{\Gamma(-\nu)}\, e^{i(\nu+1/2)\pi/2}$$
$$\times\, k^{-3/2} \left(\frac{k}{a_1\, H_1}\right)^{-1-\nu} \left(\frac{k}{a_2\, H_2}\right)^{3/2} (H_1\, H_2)^{1/2} + \cdots. \tag{6.8}$$

In eqns. (6.7) and (6.8) i is either $+$ or $\times$.

6.2 Power Spectra

During "reheating" the power spectrum of gauge-invariant fractional energy density perturbations is

$$P_{\Delta(R)}(k,t) \equiv \frac{\left\langle \left[\rho_{b\Phi(R)}\, \Delta_{\Phi(R)}\right] \left[\rho_{b\Phi(R)}\, \Delta_{\Phi(R)}\right]^* \right\rangle}{\left[\rho_{b\Phi(R)}\right]^2}$$
$$= \frac{16\pi}{m_p^2}\, k^{2(1-\nu)} \left[|A_R|^2\, (a_R)^2 + |B_R|\, (a_R)^{-1/2} + |C_R|^2\, (a_R)^{-3}\right], \tag{6.9}$$

where we have defined

$$|A_R| = \frac{1}{5} \left(\frac{\pi}{q}\right)^{1/2} (2-q)^{\nu+1/2} \frac{\csc(\nu\pi)}{\Gamma(-\nu)}\, (a_1\, H_1)^{\nu-7/2}\, (H_1)^2, \tag{6.10}$$

$$|B_R| = -\frac{4}{75}\,\frac{\pi}{q}\,\left(\frac{3-q}{2+q}\right)(2-q)^{2\nu+1}\,\frac{\csc^2(\nu\pi)}{\Gamma^2(-\nu)}\,(a_1\,H_1)^{2\nu-9/2}\,(H_1)^{3/2}, \tag{6.11}$$

$$|C_R| = \frac{2}{15}\left(\frac{\pi}{q}\right)^{1/2}\left(\frac{3-q}{2+q}\right)(2-q)^{\nu+1/2}\,\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\,(a_1\,H_1)^{\nu-1}\,(H_1)^{-1/2}, \tag{6.12}$$

where q ranges from 0 to 2 and the corresponding range of the power spectrum index $n \equiv 2(1-\nu)$ is from 1 to $-\infty$. In deriving eqn. (6.9) we have retained only the order m terms in eqn. (5.43) and used the large-scale asymptotic expansion of eqn. (5.46) [and not that of eqn. (5.33)]. It is pleasing to note that, as expected, the scale factor dependence of the right hand side of eqn. (6.9) is exactly what one would get in a pressureless fluid model (see eqn. (5.10) of Ref. [1]). The term proportional to $|A_R|^2$ in eqn. (6.9) is the growing gauge-invariant adiabatic solution (the other terms are also gauge-invariant).

The spatial momentum space two-point correlation function for gauge-invariant fractional energy density inhomogeneities during kinetic energy dominance is

$$P_{\Delta(K)}(k,t) \equiv \left\langle \Delta_{\Phi(K)}\,\Delta^*_{\Phi(K)}\right\rangle = \frac{16\pi}{m_p^2}\,k^{-2\nu}\,|\bar{A}_K|^2\,\left[J_1\!\left(\frac{1}{2}\frac{k}{a_K H_K}\right)\right]^2 (a_K)^4, \tag{6.13}$$

where J_1 is a Bessel function and

$$|\bar{A}_K| = \left(\frac{\pi}{q}\right)^{1/2}(2-q)^{\nu+1/2}\,\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\,(a_1\,H_1)^{\nu-5/2}\,(a_2\,H_2)^{-1}\,(H_1)^{5/3}\,(H_2)^{4/3}. \tag{6.14}$$

For perturbations outside the Hubble radius during kinetic energy dominance, the dominant large-scale ($k \ll a_K\,H_K$) part of eqn. (6.13) is

$$P_{\Delta(K)}(k,t) = \frac{16\pi}{m_p^2}\,k^{2(1-\nu)}\,|A_K|^2\,(a_K)^8, \tag{6.15}$$

where we have defined

$$|A_K| = \frac{1}{4}\left(\frac{\pi}{q}\right)^{1/2}(2-q)^{\nu+1/2}\,\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\,(a_1\,H_1)^{\nu-5/2}\,(a_2\,H_2)^{-4}\,(H_1)^{5/3}\,(H_2)^{10/3}; \tag{6.16}$$

we note that the scale factor dependence of eqn. (6.15) is that expected for the dominant large-scale adiabatic solution during kinetic energy dominance [see eqn. (3.104)]. We also note, from the discussion following eqn. (3.107), that the adiabatic contribution to the fractional energy density power spectrum is given by $\left\langle \delta_{\Phi(K)}\,\delta^*_{\Phi(K)}\right\rangle = (16/49)\,P_{\Delta(K)}$.

It is interesting to note that eqns. (6.9) and (6.15) have an identical dependence on spatial momentum; this dependence also coincides with that of the baryon dominated epoch power spectrum in the model of Refs. [7,1] [eqn. (5.10) of Ref. [1]; the model of Ref. [1] also has three epochs: an early inflation epoch (which is modelled in the same manner as in this paper) and the more conventional late time radiation and baryon dominated epochs] — we conclude that the spatial momentum dependence of the large-scale late time fractional energy density power spectrum is completely determined by the inflation epoch model (and is independent of the model used to describe the late time universe).

We shall have need for the spatial momentum space form of the temporal displacement of the spatially homogeneous scalar field spatial hypersurface from the corresponding constant time synchronous gauge spatial hypersurface, eqn. (4.32),

$$\Delta t(t,k) = -\frac{\phi(t,k)}{\dot{\Phi}_b(t)} = -i\,\frac{a}{k}\,v_\Phi^1(t,k), \tag{6.17}$$

where v_Φ^1 is the longitudinal peculiar velocity and the second equation follows from eqns. (3.42), (3.43) and (3.46). Using the results of Ref. [1], we find, during inflation,

$$\Delta t_I(t,k) = -\frac{1}{2}\sqrt{\frac{q}{2}}\,c_2 - \left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{1}{4}\left(\frac{2\pi}{q}\right)^{1/2} e^{i(\nu-1/2)\pi/2}\,\frac{k^{-1/2}}{a_I\,H_I}\,G_1(x_I), \tag{6.18}$$

where G_1 is given in eqn. (5.38) with

$$\lambda\, e^{-(2-q)x_I/q} = \frac{2}{2-q}\,\frac{k}{a_I\,H_I}. \tag{6.19}$$

The first term on the right hand side of eqn. (6.18) is the contribution from the inflation epoch gauge-dependent time translation solution while the second term includes the gauge-invariant adiabatic solution. During “reheating” we find [from eqns. (5.4), (5.5), (5.18), (5.21) and (5.27) – (5.29)] that the temporal displacement (to leading order in m^{-1}) is

$$\begin{aligned}\Delta t_R(t,k) = &\left[\sqrt{q}\,\cos\{m(t-t_1)\} - \sqrt{6-q}\,\sin\{m(t-t_1)\}\right]^{-1}\\ &\times\Bigg[\left(C_1\,a_R + C_2 + \frac{C_4}{(a_R)^{3/2}}\right)\sin\{m(t-t_1)\}\end{aligned}$$

$$+\sqrt{\frac{6-q}{q}}\left(C_1\,a_R-\frac{q}{6-q}\,C_2+\frac{C_4}{(a_R)^{3/2}}\right)\cos\{m(t-t_1)\}\Bigg], \qquad (6.20)$$

where we have defined

$$C_1=-\frac{1}{\sqrt{2}}\left(\frac{3}{2}\right)^{1/3}(H_1)^{-2/3}\,(a_1)^{-1}\,c_1^{(R)}, \qquad (6.21)$$

$$C_2=-\frac{3}{2\sqrt{2}}\,c_2^{(R)}, \qquad (6.22)$$

$$C_4=-\frac{9}{4\sqrt{2}}\,H_1\,(a_1)^{3/2}\,c_4^{(R)}, \qquad (6.23)$$

and $c_1^{(R)}$, $c_2^{(R)}$ and $c_4^{(R)}$ are the "reheating" epoch constants of integration that appear in eqns. (5.28) and (5.29). From eqn. (3.51) we find, during kinetic energy dominance,

$$\Delta t_K(t,k)=-\frac{\sqrt{3}}{2}\,c_2^{(K)}+\frac{1}{3^{1/6}\sqrt{2}}\;k^{-3/2}\,\frac{(a_K)^{3/2}}{(H_K)^{1/6}}\left(c_+^{(K)}\,G_1^K(x_K)+c_-^{(K)}\,G_2^K(x_K)\right), \qquad (6.24)$$

where G_1^K and G_2^K are given by eqns. (3.21) and (3.22) with

$$\lambda_K\,e^{2x_K/3}=\frac{1}{2}\frac{k}{a_K\,H_K}. \qquad (6.25)$$

We now turn to the two-point characterization of gravitational wave perturbations. From eqns. (5.50) and (6.7) we find, during "reheating",

$$\left\langle h_i^{(R)}\,h_i^{(R)*}\right\rangle=\frac{16\pi}{m_p^2}\,k^{-2(\nu+4)}\,(a_R)^{-3}\,|\bar{D}_R|^2 \qquad (6.26)$$
$$\times\left[1-\cos\left(\frac{4k}{a_RH_R}\right)-\frac{4k}{a_RH_R}\sin\left(\frac{4k}{a_RH_R}\right)+\left(\frac{2k}{a_RH_R}\right)^2\left\{1+\cos\left(\frac{4k}{a_RH_R}\right)\right\}\right],$$

where i is either $+$ or $\times$ [there is no implicit summation over i on the left hand side of this equation (and eqns. (6.28), (6.30) and (6.32) below)] and

$$|\bar{D}_R|=\frac{3}{8}\sqrt{\pi}\,(2-q)^{\nu+1/2}\,\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\,(a_1\,H_1)^{\nu+4}\,(H_1)^{-1/2}. \qquad (6.27)$$

We note that both the spatial momentum dependence as well as the temporal dependence of eqn. (6.26) coincides with that of eqn. (5.21) of Ref. [1] (the corresponding baryon dominated epoch expression in that model). For large-scale perturbations eqn. (6.26) reduces to

$$\left\langle h_i^{(R)}\, h_i^{(R)*} \right\rangle = \frac{16\pi}{m_p^2}\, k^{-2(1+\nu)}\, |D_R|^2 \tag{6.28}$$

where we have defined

$$|D_R| = \sqrt{2\pi}\,(2-q)^{\nu+1/2}\,\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\,(a_1\,H_1)^{\nu-1/2}\,H_1. \tag{6.29}$$

Similarly, from eqns. (3.34) and (6.8), we find, for gravitational wave inhomogeneities during kinetic energy dominance,

$$\left\langle h_i^{(K)}\, h_i^{(K)*} \right\rangle = \frac{16\pi}{m_p^2}\, k^{-2(1+\nu)} \left[J_0\left(\frac{1}{2}\frac{k}{a_K H_K}\right)\right]^2 |D_K|^2, \tag{6.30}$$

where J_0 is a Bessel function and

$$|D_K| = \sqrt{2\pi}\,(2-q)^{\nu+1/2}\,\frac{\csc(\nu\pi)}{\Gamma(-\nu)}\,(a_1\,H_1)^{\nu+1}\,(a_2\,H_2)^{-3/2}\,(H_1\,H_2)^{1/2}, \tag{6.31}$$

or on large scales,

$$\left\langle h_i^{(K)}\, h_i^{(K)*} \right\rangle = \frac{16\pi}{m_p^2}\, k^{-2(1+\nu)} |D_K|^2. \tag{6.32}$$

The spatial momentum dependence and the temporal dependence of the large-scale part of the gravitational wave two-point functions during "reheating" and kinetic energy dominance, eqns. (6.28) and (6.32), coincide (and is the same as that during baryon dominance, eqn. (5.23), in the model of Ref. [1]) — we conclude that the spatial momentum dependence of this large-scale late time two-point function is fixed by the inflation epoch model.

6.3 de Sitter Inflation Limit

In the limit $q = 2\epsilon^2 \to 0$ the inflation epoch model used here reduces to the de Sitter inflation model studied in Refs. [5,1]. In this limit we find that the gauge-invariant fractional energy density power spectrum during "reheating", eqn. (6.9), reduces to

$$P_{\Delta(R)}(k,t) = \frac{1}{\epsilon^2}\,\frac{16\pi}{m_p^2}\,k\left[|\hat{A}_R|^2\,(a_R)^2 + |\hat{B}_R|\,(a_R)^{-1/2} + |\hat{C}_R|^2\,(a_R)^{-3}\right], \tag{6.33}$$

where we have defined

$$|\hat{A}_R| = \frac{1}{5\sqrt{2}}\,(a_1)^{-3}\,(H_1)^{-1}, \tag{6.34}$$

$$|\hat{B}_R| = -\frac{1}{25}\,(a_1)^{-7/2}\,(H_1)^{-2}, \tag{6.35}$$

$$|\hat{C}_R| = \frac{1}{5\sqrt{2}}\,(a_1)^{-1/2}\,(H_1)^{-1}, \tag{6.36}$$

while the large-scale part of that during kinetic energy dominance, eqn. (6.15), is given by

$$P_{\Delta(K)}(k,t) = \frac{1}{\epsilon^2}\,\frac{16\pi}{m_p^2}\,k\,|\hat{A}_K|^2\,(a_K)^8, \tag{6.37}$$

where

$$|\hat{A}_K| = \frac{1}{4\sqrt{2}}\,(a_1\,H_1)^{-2}\,(a_2\,H_2)^{-4}\,(H_1)^{5/3}\,(H_2)^{10/3}. \tag{6.38}$$

It is pleasing to note that eqns. (6.33) and (6.37) are of the scale-invariant form ($\propto k$), [16], as is expected of de Sitter inflation, [1]. Once again, [1,3], the large time energy density power spectra, eqns. (6.33) and (6.37), "diverge" in the limit of de Sitter inflation. (More precisely, since the linear perturbation approximation is no longer valid for very small ϵ, we may only conclude that the power spectra get larger as ϵ is decreased.)

The physical reason for the "divergent" late time power spectra in this model (in the de Sitter inflation limit) seems to be similar to that responsible for the de Sitter inflation limit large time "divergences" in the models of Refs. [1,3] — a "slowly rolling", inflation epoch, spatially homogeneous scalar field. The situation in the model at hand (during "reheating") is somewhat more complicated than that encountered in the models of Refs. [1,3] (and, during "reheating", we will only provide a partial justification for the connection between a "slowly rolling", inflation epoch, background scalar field and the "divergence" of the late time power spectrum). More specifically, in Refs. [1,3] we found, during inflation, that the temporal displacement of the spatially homogeneous energy density hypersurface from the synchronous gauge constant time hypersurface "diverged" in the de Sitter limit. (The displacement, Δt, is proportional to the fractional energy density irregularity, but the inflation epoch "divergence" is mostly due to a time derivative of the background energy density, $\Delta t \propto \delta_\Phi / \dot{\rho}_{b\Phi}$.) Since we join the perturbations on spatially homogeneous energy density hypersurfaces, the inflation epoch "divergence" in Δt is carried through to large times (the late time power spectrum also "diverges"). A more detailed version of this description may be found in Refs. [1,3] — here we underscore that, with the joining conditions used there, Δt is proportional to the fractional energy density irregularity, δ_Φ. The first reason that makes it more involved to conclude here that the "slowly rolling" field is responsible for the ϵ^{-2} "divergence" in eqns. (6.33) and (6.37) is that (for the spatially homogeneous scalar field hypersurface joining conditions used here) the temporal displacement is not directly related to δ_Φ [it is related to the longitudinal peculiar velocity, eqn. (6.17)]. The second complication is that the analysis of the "reheating" epoch, [13], makes use of a double perturbation expansion (in spatial irregularity and $1/m$), unlike the case of the radiation or baryon epoch analyses of Refs. [1,3] where we only use an expansion in spatial irregularity.

These complications notwithstanding, it proves illuminating to examine the behaviour of the temporal displacement of the spatially homogeneous scalar field hypersurface from the synchronous gauge constant time hypersurface, eqns. (6.18), (6.20) and (6.24), in the de Sitter inflation limit. From eqn. (6.18), we find, in this limit, during inflation,

$$\Delta t_I(t,k) = -\frac{\epsilon}{2}\, c_2 - \frac{i}{\epsilon}\left(\frac{16\pi}{m_p^2}\right)^{1/2} (2k)^{-3/2} \left(1 - i\frac{k}{a_I H_I}\right) \exp\left(i\frac{k}{a_I H_I}\right), \tag{6.39}$$

where, using eqns. (5.24) – (5.26), we may write

$$a_I H_I = a_{\rm now} H_{\rm now}\, \frac{(1+z_2)^{3/2}(1+z_1)^{3/2}}{(1+z)}, \tag{6.40}$$

which does not depend on ϵ. Since z_1 and z_2 are free parameters, eqn. (6.39) implies that for small ϵ (ignoring the gauge-dependent time translation solution contribution) Δt_I [and, from the second of eqns. (6.17), $v^1_{\Phi(I)}$] "diverges" like ϵ^{-1}. (This "divergence" does not get reflected in the inflation epoch energy density power spectrum, see eqn. (3.56) of Ref. [1].) We emphasize that the "divergence" is entirely due to the denominator of the first of eqns. (6.17) [since, from eqn. (2.7) of Ref. [1], $\dot{\Phi}_{bI}(t) = 2\,\epsilon\,H_1$ and it may be shown that the adiabatic part of ϕ_I is of order ϵ^0 (see Sec. 5.2 of Ref. [1] — this means that the standard assumption of linear perturbation theory is not violated during de Sitter inflation, even though Δt_I gets large)]. The evolution of the background scalar field, during de Sitter inflation, is so slow that one is forced to go to a synchronous gauge hypersurface of very different time before Φ_{bI} changes sufficiently to soak up the spatially inhomogeneous part of the scalar field, ϕ_I. It is interesting to note that the leading large-scale term in eqn. (6.39) coincides with that in the expression for the temporal displacement to the spatially homogeneous energy density spatial hypersurface, eqn. (5.20) of Ref. [1].

From eqns. (5.5), (5.18) and (5.21) we find (in the de Sitter inflation limit), at the "reheating" side of the first transition (to lowest order in m^{-1}),

$$\dot{\Phi}_{bR}(t_1) = 2\,\epsilon\,H_1 \tag{6.41}$$

[since we have made use of an expansion in $1/m$ to solve the "reheating" model, the next term in this equation will be suppressed by a factor of H_1/m — more precisely, for eqn. (6.41) to be valid we must require $\epsilon \gg (3\sqrt{3}/2)\,(H_1/m)$]. At the first transition, the "reheating" temporal displacement, eqn. (6.20), in the de Sitter inflation limit, becomes

$$\Delta t_R(t_1, k) = \frac{1}{\epsilon^2}\sqrt{\frac{3}{2}}\left(C_1\,a_R + \frac{C_4}{(a_R)^{3/2}}\right) - \frac{1}{\sqrt{6}}\,C_2. \tag{6.42}$$

To be able to join this onto Δt_I at t_1, one might be tempted to conclude, from eqns. (6.21) – (6.23), that it is necessary to have $c_1^{(R)}$ and $c_4^{(R)}$ be of order ϵ and $c_2^{(R)}$ be of order ϵ^{-1}. This is only partially correct — the (inflation epoch) adiabatic part of $c_2^{(R)}$ is of order ϵ^{-1} [eqn. (6.2)] while the (inflation epoch) adiabatic terms in eqns. (6.1) and (6.3) imply that $c_1^{(R)}$ and $c_4^{(R)}$ are of order ϵ^0. The a posteriori reason for this discrepancy is that the adiabatic

$c_2^{(R)}$ contribution to eqn. (6.42) is of order m^0 while the other contributions are down by a factor of $1/m$ [see eqns. (6.1) – (6.3) and (6.21) – (6.23)] — we conclude that the order (in m^{-1}) to which we have consistently worked prevents us from determining, from dimensional analysis alone, the ϵ dependence of C_1 and C_4. In any case, that C_2 is of order ϵ^{-1} allows us to conclude that $\Delta_{\Phi(R)}$ has a contribution of order ϵ^{-1} [from eqn. (5.43) and the fact that we used the large-scale expansion of eqn. (5.46) to derive eqn (6.9)], and so the expression for $P_{\Delta(R)}$ must have a term of order ϵ^{-2} (in the de Sitter inflation limit), as we find in eqn. (6.33). [We note that during "reheating" the (inflation epoch) adiabatic part of ϕ_R and $v^1_{\Phi(R)}$ are of order ϵ^{-1}, so when ϵ is small ϕ_R is large and the standard assumption of linear perturbation theory is no longer valid.]

Turning now to the second transition, we find, in the de Sitter inflation limit,

$$\dot{\Phi}_{bR}(t_2) = 2\sqrt{3}\, H_2, \tag{6.43}$$

(which does not decrease as ϵ becomes smaller — this is a consequence of the conversion of the scalar field potential energy to kinetic energy during "reheating") and

$$\Delta t_R(t_2, k) = -\frac{1}{\sqrt{6}}\, C_2 \propto \epsilon^{-1}, \tag{6.44}$$

where the second relation follows from the dimensional analysis of the previous paragraph. For eqn. (6.24) (at t_2) to agree with eqn. (6.44) we must require that $c_2^{(K)}$ and $c_\pm^{(K)}$ be of order ϵ^{-1} [which is true of the adiabatic contribution to eqn. (6.5) and of eqn. (6.6)] — as a result, $\Delta_{\Phi(K)}$ is of order ϵ^{-1} [eqn. (3.76)] and in the de Sitter inflation limit $P_{\Delta(K)}$ is of order ϵ^{-2} [which agrees with eqn (6.37)]. (As during "reheating", during kinetic energy dominance the adiabatic part of ϕ_K and $v^1_{\Phi(K)}$ are of order ϵ^{-1}.)

In summary, we have provided partial physical justification for holding the "slowly rolling" background scalar field during the de Sitter inflation epoch responsible for the ϵ^{-2} "divergence" in the late time large-scale energy density power spectra.

To conclude this section we examine the large-scale late time gravitational wave two-point functions in the de Sitter inflation limit. We find that the "reheating" epoch gravitational wave perturbation two-point function, eqn. (6.28), reduces to

$$\left\langle h_i^{(R)}\, h_i^{(R)*}\right\rangle = \frac{16\pi}{m_p^2}\, k^{-3}\, |D_R|^2 \tag{6.45}$$

[there is no implied summation over i, $= +$ or $\times$, on the left hand side of this equation and eqn. (6.47) below], where we have defined

$$|D_R| = \sqrt{2}\, H_1, \tag{6.46}$$

while that during kinetic energy dominance, eqn. (6.32), is given by

$$\left\langle h_i^{(K)}\, h_i^{(K)*}\right\rangle = \frac{16\pi}{m_p^2}\, k^{-3} |D_K|^2, \tag{6.47}$$

where

$$|D_K| = \sqrt{2}\, (a_1\, H_1)^{3/2}\, (a_2\, H_2)^{-3/2}\, (H_1\, H_2)^{1/2}. \tag{6.48}$$

Unlike the fractional energy density power spectra, the gravitational wave perturbation two-point functions do not "diverge" in the exponential expansion inflation limit.

7. Conclusion

We have designed a simple, scalar field dominated, cosmology which inflates and "reheats" and have used it to study the evolution of inflation generated spatial irregularities. The transitions in this model are less complicated than in models we have previously used and the joining conditions at the transitions also differ from our earlier joining conditions.

Two interesting results of our analysis are: the spatial momentum dependence of the late time large-scale two-point correlation functions of energy density irregularities and gravitational wave perturbations are determined by the inflation model used (including the quantum mechanical initial conditions) and seem to be independent of the manner in which the postinflation universe is modelled (which determines the temporal dependence of these statistics); and, the late time energy density power spectrum "diverges" in the de Sitter inflation limit. The second result supports the conclusions of Refs. [1,3] (that it is premature to claim that the inflation scenario of the very early universe has a "fine-tuning" problem).

Even though the main features of the large-scale late time two-point functions do not seem to depend sensitively on the manner in which we approximate "reheating" (i.e., the prescription of Refs. [1,3] and that used here lead to similar results), we emphasize that our results should be considered to be preliminary since neither of these prescriptions for "reheating" has been shown to follow from microphysical considerations. This is one of the aspects of the inflation scenario that we hope to return to in due course of time.

Acknowledgements

I am indebted to J. Peebles and L. Susskind for valuable advice. I also acknowledge useful discussions with J. Bardeen, R. Brandenberger, J. Hwang, D. Mitchell, M. Peskin, S. Phinney, J. Preskill, C. Thompson and R. Wagoner. This work was supported in part by the NSF, grant AST84-51725, by the DOE, contract DE-AC03-81ER40050, and by the California Institute of Technology.

References

1. B. Ratra, Phys. Rev. **D45**, 1913 (1992).
2. B. Ratra, Phys. Rev. **D40**, 3939 (1989).
3. B. Ratra, Caltech preprint GRP-217/CALT-68-1594 (1989).
4. B. Ratra, Phys. Rev. **D31**, 1931 (1985).
5. W. Fischler, B. Ratra and L. Susskind, Nucl. Phys. **B259**, 730 (1985).
6. B. Ratra, Phys. Rev. **D43**, 3802 (1991).

7. F. Lucchin and S. Matarrese, Phys. Rev. **D32**, 1316 (1985);
J. J. Halliwell, Phys. Lett. **185B**, 341 (1987).

8. B. Ratra and P. J. E. Peebles, Phys. Rev. **D37**, 3406 (1988).

9. B. Ratra, Phys. Rev. **D44**, 365 (1991).

10. V. Müller, H. J. Schmidt and A. A. Starobinsky, Class. Quant. Grav. **7**, 1163 (1990);
S. Matarrese, F. Lucchin and A. Ortolan, Padova preprint DFPD 89/A/70 (1989);
J. D. Barrow and K. Maeda, Nucl. Phys. **B341**, 294 (1990);
A. L. Berkin, K. Maeda and J. Yokoyama, Phys. Rev. Lett. **65**, 141 (1990).

11. S. W. Hawking, Phys. Lett. **115B**, 295 (1982);
A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982);
A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982);
J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. **D28**, 679 (1983).

12. B. Ratra, Phys. Lett. **B260**, 21 (1991).

13. B. Ratra, Phys. Rev. **D44**, 352 (1991).

14. B. Ratra, Phys. Rev. **D38**, 2399 (1988).

15. J.-c. Hwang and E. T. Vishniac, Ap. J. **353**, 1 (1990);
D. H. Lyth and E. D. Stewart, Ap. J. **361**, 343 (1990);
V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. **215**, 203 (1992), and references therein.

16. E. R. Harrison, Phys. Rev. **D1**, 2726 (1970);
P. J. E. Peebles and J. T. Yu, Ap. J. **162**, 815 (1970);
Ya. B. Zel'dovich, Mon. Not. R. Astr. Soc. **160**, 1P (1972).

17. R. H. Dicke, Ap. J. **152**, 1 (1968);
A. Vilenkin, Phys. Rev. Lett. **48**, 59 (1982).

18. Ya. B. Zel'dovich, Sov. Phys. J.E.T.P. **14**, 1143 (1962).