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Inflation in a Quartic Potential Scalar Field Model

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ABSTRACT

A spatially flat cosmological scalar field (Φ) model with the scalar field potential $V(\Phi) = 2\Lambda\left[1 - \epsilon\,(\Phi - \bar{\Phi})^4/4\right]$, where Λ is the positive cosmological constant during inflation, ϵ is a small positive parameter and $\bar{\Phi}$ is the minimum value of the scalar field for which this approximate expression for the potential holds, has been extensively used as a simple approximation to a more complicated (GUT inspired) very early universe model cosmology that inflates. We study this model, using techniques we have previously developed, augmented with a perturbation expansion in ϵ, to estimate the large time, baryon dominated epoch, form of the spectrum of energy density irregularities, of local departure velocity from homogeneous expansion, and of gravitational wave energy density. The magnitude of the power spectrum of energy density irregularities depends on the initial values of the spatially homogeneous background part of the scalar field and its time derivative, which are free parameters in the context of this approximate model. We argue that the dynamical behaviour of the model is not qualitatively sensitive to the specific values chosen for these parameters (under some restrictions) and derive expressions for spatial irregularities using a partially specified set of simplifying initial conditions for the background scalar field. The explicit dependence of the power spectrum on ϵ agrees with our earlier conjecture and differs from what was previously found by other authors. This difference is a consequence of the difference between the "reheating" prescription used here and that used in earlier analyses.

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1. Introduction and Summary

Quantum mechanical fluctuations during a very early epoch of inflation could generate the small spatial irregularities that have to be assumed as initial conditions in the canonical gravitational instability model for the formation of structure in the universe. To examine this possibility we have analyzed the generation and evolution of small spatial inhomogeneities in a simple class of scalar field dominated inflation modified hot big bang models, [1,2]. The evolution of a spatially flat cosmological model is partitioned into three epochs: an early scalar field dominated inflation epoch, during which the expansion of the universe is driven by a scalar field (Φ) with an exponential potential, $V(\Phi) \propto \exp(-\Phi/\sqrt{p})$, $p > 1$ (this results in a scale factor $a(t) \propto t^p$, see Sec. 3.1 of Ref. [3]); an intermediate radiation dominated epoch (the scalar field potential is set to zero at the onset of radiation dominance); and the present baryon dominated ($\Omega_{\rm now} = 1$) epoch. [The density parameter, Ω, is the ratio of the mean mass density to the Einstein-de Sitter value, $\Omega(t) = 8\pi G\rho/(3H^2)$, where G is the gravitational Newton constant, H is the Hubble parameter and ρ the relevant energy density; $\Omega_{\rm now}$ denotes the present value of the density parameter.]

In each epoch we only allow for perturbations in the dominant form of matter. We also assume that the transitions between epochs are instantaneous, i.e., we require that the equation of state (and hence the pressure) change instantaneously and discontinuously on the spatial hypersurface at which the local (total) energy density reaches an appropriate value. We emphasize that since the transitions are governed by local physics, this macrophysical prescription — at least for the scalar field dominance to radiation dominance transition — will have to be verified by a microphysical calculation. (For some justification and for a more detailed description of the instantaneous transition model see Refs. [1,4].) In each epoch we derive closed-form general solutions of the synchronous gauge relativistic linear perturbation equations which govern the evolution of inhomogeneities, using the techniques of Ref. [5]. The constants of integration in the inflation epoch expressions are determined from the quantum mechanical initial conditions of Ref. [6], while those in the radiation and baryon dominated epoch expressions are determined from the joining conditions of Ref. [4].

We find that in the limit $q = 2\epsilon^2 \sim 0$ (where $q = 2/p$) the inflation epoch scalar field potential used in Refs. [1,2] reduces to the step-function potential of Ref. [7],

$$V(\Phi) = 2\Lambda[1 - \epsilon\,(\Phi - \Phi^{(0)})], \tag{1.1}$$

where ϵ is a positive parameter, Λ is the positive cosmological constant during inflation and $\Phi^{(0)}$ is a constant — this limit corresponds to exponential expansion (or de Sitter) inflation. In this limit the large time power spectrum of large-scale, gauge-invariant, adiabatic energy density irregularities,

$$P(k,t) \equiv \langle \delta_B(k,t)\delta_B^*(k,t)\rangle = \frac{16\pi}{m_p^2}\, k^{2(1-\nu)}|A|^2 a^2 \tag{1.2}$$

(here δ_B is the fractional baryonic energy density perturbation, k is the magnitude of coordinate spatial momentum, $\nu = (2+q)/[2(2-q)], m_p = G^{-1/2}$ is the Planck mass, a is the FLRW scale factor and $|A|$ is a numerical factor which depends on the time of "reheating" and the time of the radiation–baryon transition and is defined in Sec. 5.2 of Ref. [1]), reduces to the scale-invariant form, [8],

$$P(k,t) \propto \frac{1}{\epsilon^2}\, ka^2, \tag{1.3}$$

where the numerical prefactor increases as the potential, eqn. (1.1), becomes flatter and as the model more closely approaches exact de Sitter inflation. The geometrical reason for this "divergent" behavior is discussed in Sec. 5.2 of Ref. [1] (and is reviewed in Sec. 3.2 below). The scale invariance of the power spectrum of adiabatic fluctuations is a consequence of the form of the potential of the inflaton field — a very flat inflaton potential results in an exponentially expanding scale factor and a scalar field two-point function that varies logarithmically with scale, [6]; this results in a scale-invariant power spectrum, [7].

Since the inflation scenario is, at present, only a model (i.e., it is not a parameter free theory of the very early universe), it is only possible to examine the constraints which large time observations impose on the free parameters of the model. In Ref. [1] we found that observational data does not unduly constrain the exponential potential inflation model, even in the de Sitter inflation limit. In particular, models which stop inflating around 10^7 — 10^{16} GeV (a given numerical value is tied to the spectral index of the power spectrum of energy density irregularities) are not obviously inconsistent with large-scale observational data. Furthermore, this consistency does not require that a parameter assume a value of $\lesssim 10^{-10}$, as has been widely claimed to be a necessary prerequisite of observationally consistent inflation models, [9-12].

These authors, among very many others, have studied an exponential expansion inflation model with the scalar field potential

$$V(\Phi) = 2\Lambda \left[1 - \frac{1}{4}\epsilon \, (\Phi - \bar{\Phi})^4\right], \tag{1.4}$$

where ϵ is a small positive parameter and $\bar{\Phi}$ (≥ 0) is the minimum value of the scalar field for which this approximate (GUT inspired) expression for the potential holds. They find that the large time power spectrum of energy density irregularities is

$$P(k) \propto \epsilon \, k \ln^3(k); \tag{1.5}$$

they also find that the power spectrum is almost independent of the time of "reheating" and argue that large-scale observational data forces ϵ to be exceedingly small. These results differ from what we have found for exponential expansion inflation. In particular, in the limiting de Sitter model, eqn. (1.1), when ϵ is decreased the potential becomes flatter, the model more closely approximates exact exponential expansion inflation and the power spectrum increases, eqn (1.3). In the model described by eqn. (1.4), when ϵ is decreased the potential also becomes flatter and the model also more closely approximates exact exponential expansion inflation, yet the power spectrum decreases, eqn. (1.5). This different limiting behaviour is the reason why previous analyses have concluded that ϵ needs to be exceedingly small. We also note that the spatial momentum dependence of the power spectrum found in earlier analyses differs from what we have found.

Could it be that these differences are just a reflection of the different forms of the approximate, inflation epoch, potentials of the two models, eqns. (1.1) and (1.4)? No, because the "divergence" (for small ϵ) of the amplitude in eqn. (1.3) is a consequence of a geometric property of the background de Sitter spacetime (discussed in Sec. 3.2 below); since the model described by eqn. (1.4) is also a de Sitter inflation model, the power spectrum must, as shown below, exhibit this "divergence". The reason — discussed in Refs. [13,14], and summarized below — for these differences is the difference in the prescriptions used to implement "reheating" (in Refs. [9-12] and in Ref. [1]).

The main purpose of this paper is to study the inflation modified hot big bang cosmological model characterized by the potential given in eqn. (1.4), using the methods and approximations of Refs. [1,5,4]. In particular, we wish to determine the ϵ dependence of the energy density power spectrum in order to see if it agrees with the result of Ref. [1] (if we make use of the approximations used there, even though the scalar field potentials differ) and to determine whether observational data forces ϵ to be exceedingly small as has been claimed in earlier analyses. Interestingly, we do not completely succeed: the explicit ϵ dependence of the large time power spectrum is $\propto \epsilon^{-2}$ (which agrees with the conjecture of Ref. [1] and differs from the results of all other analyses), however, the power spectrum also depends on the initial values of the background scalar field and its time derivative at the "beginning" of inflation, initial values that cannot be determined from the model described by the approximate potential (1.4). (We argue, in the Appendix, that the dynamics is not qualitatively sensitive to these initial values, provided a certain restriction is satisfied, and estimate the spatial irregularities using a simplifying, partial, choice of initial values.) It is therefore not true that observational data forces a parameter, ϵ, of the inflation epoch potential to assume an exceedingly small value, even though we have not completely succeeded in comparing this inflation model to observational data (because of the dependence on undetermined initial conditions). There is, however, a limit in which the quartic potential inflation model reduces to the limiting de Sitter inflation model studied in Ref. [1]; in this limit the quartic potential inflation model does not seem to be unduly constrained by observational data. (The "nonlinear" scalar field potential of the model studied in Refs. [1,7] results in a time-dependent fixed point solution for the spatially homogeneous background fields, [3], and ensures that the large time spatial irregularities are independent of the background scalar field initial values.)

The general procedure we use to analyze the quartic potential inflation model is described in Ref. [1]. Here, and in the introduction to Sec. 3, we comment on a few aspects of this procedure. The single "fluid" instantaneous transition model is based on the assumption that, for large-scale purposes, it suffices to focus on the macrophysics of the inflation modified hot big bang model and to ignore the microphysics. We would like to determine the large time, baryon dominated epoch, characteristics of the inflation scenario; the instantaneous transition model is a physically motivated approximate model that may be used to make this problem analytically tractable.

In the single "fluid" instantaneous transition model one only accounts for perturbations in the dominant form of matter in each epoch — since the subdominant forms of matter are ignored, joining conditions are needed to match the perturbations at the transition. These joining conditions are derived by requiring that the perturbation equations of motion be singularity-free at the transition spatial hypersurface, [4], which is taken to be a spatially homogeneous local energy density hypersurface (since the transitions are governed by local physics). We note that these joining conditions differ from those used in earlier analyses, [4]. When we use these joining conditions we do not, as is commonly done, approximate our expressions for the perturbations by retaining only the low spatial momentum part;

we use the complete solutions to determine the constants of integration in the expressions after a transition and then approximate the expressions for the constants of integration by focussing on spatial momentum modes that were on scales larger than the Hubble scale at the transition. These large-scale perturbations are truly characteristic of inflation, since they have not been processed by small-scale physics. It is well known that if one accounts for the small-scale physics which describes the coupling of radiation to matter, the only significant effect is a modification of the small-scale part of the power spectrum — the large-scale form of the power spectrum is unaffected, [15]. One might reasonably expect the small-scale physics of the reheating transition to have a not very dissimilar effect, provided the "reheating" transition occurs rapidly on the relevant Hubble timescale.

One point to be noted about the model described by eqn. (1.4) is that if ϵ were large one would not have de Sitter inflation. Since ϵ must be small during de Sitter inflation we shall make use of a double perturbation expansion — in the Newton constant (spatial irregularity) and in ϵ (scalar field nonlinearity) — and work to lowest nontrivial order in both expansions. (It would be of interest to determine whether the large ϵ model also inflates.)

Using the techniques of Refs. [1,5,4] we evaluate the power spectrum of energy density irregularities, the spectrum of local departure velocity from homogeneous expansion and the gravitational wave energy density spectrum in the large time, baryon dominated, universe. [We note that the perturbations of interest are those that have entered the Hubble radius in the recent past and those that are entering now, however, unlike as in previous analyses, we do not have to determine when these perturbations left the Hubble radius during inflation since this is taken care of by our formalism.] These expressions depend on some of the free parameters of the inflation model of the very early universe. These free parameters can only be constrained by comparing the expressions for the perturbations to large-scale observational data (since the small-scale form of our expressions are not physically relevant). To compare theory to observation we must ensure that the theoretical expression and the corresponding observational measurement are recorded in the same coordinate system. We must also only compare a position space course-grain average of the theoretical expression to the observational measurement. (The general procedure is described in Sec. 5 of Ref. [1].) We shall not bother to make this comparison here since we find, in the inflation model described by the potential (1.4), that the relevant large time, large-scale, perturbations depend on one more free parameter than those of the model studied in Ref. [1] (which only depend on two free parameters), and since observational data does not unduly constrain the parameters of the exponential potential inflation model of Ref. [1], it would be somewhat surprising if it were found that consistency with the observations required that the inflation model described by eqn. (1.4) be "fine-tuned".

We finally summarize the reason for the differences between the power spectrum we have found for the quartic potential inflation model and that found in earlier analyses (a more complete discussion is given in Refs. [13,14]). In our analysis, the approximate expression for the scalar field potential, eqn. (1.4), only holds during inflation and does not determine the time at which inflation ends (which is a free parameter). In the earlier analyses, this approximate potential is also used to determine the time at which inflation ends (which is taken to be the time when the "slow-rolling" approximation breaks down), [13]. With this prescription for the end of inflation, the scalar field takes a time of the order of the appropriate Hubble time to change from the value it has at the end of inflation to the value at the minimum of the complete scalar field potential — this is the slow "reheating" prescription, i.e., the relevant timescale is the cosmological one. In our case "reheating" occurs instantaneously. The difference in the "reheating" timescales is responsible for the differences in the power spectra. (Since a model that instantaneously "reheats" is physically unrealistic, we have also looked at a model in which the "reheating" timescale is finite but much shorter than the cosmological timescale at this epoch, [16]; the de Sitter inflation limit "divergence" is still present.)

In Sec. 2 we study the evolution of the background fields (technical details are relegated to an appendix) and the evolution of small spatial irregularities during the inflation epoch of the model described by eqn. (1.4), working to lowest nontrivial order in both G and ϵ, derive solutions of the equations of motion and use the remnants of general coordinate invariance in synchronous gauge to catalogue physical solutions. We then use the quantum mechanical initial conditions of Ref. [6] to determine the constants of integration that appear in these solutions, present asymptotic approximations of the inflation epoch expressions (valid on small scales and on large scales) and tabulate the spatial momentum space two-point correlation functions of some of the fields. In Sec. 3 we use these results and the results of Ref. [5], as well as the joining conditions of Ref. [4] to derive expressions for small spatial irregularities in the radiation and matter dominated epochs. We then approximate the constants of integration in the baryon dominated epoch expressions by focussing on scales that were larger than the Hubble scale at the transition from scalar field dominance to radiation dominance and the radiation–baryon transition and derive the large time, baryon dominated epoch, form of the peculiar velocity perturbation and the fractional energy density irregularity power spectrum.

2. Inflation

The Einstein-scalar field action is

$$S = \frac{m_p^2}{16\pi}\int dt\, d^3x \sqrt{-g}\left[-R + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}V(\Phi)\right] \tag{2.1}$$

where the metric signature is $(+---)$. The inflation model we study is characterized by the scalar field potential given in eqn. (1.4), which is valid for $\Phi \geq \bar{\Phi}$. The connection between our notation and that of Refs. [9-12] is

$$\Lambda \propto (g\sigma)^4, \qquad \Lambda\epsilon \propto \lambda \propto g^4, \tag{2.2}$$

where g is the gauge coupling (of the underlying GUT model), σ is the expectation value of the scalar field at the minimum of the (more complete) potential and λ is proportional to the coefficient of the quartic scalar field term in the potential. Λ or $g\sigma$ has a fixed numerical value that is related to the GUT energy density scale. It has occasionally been claimed that it is actually σ that is held fixed and

$$\Lambda \propto \Lambda\epsilon \propto \lambda \propto g^4; \tag{2.3}$$

this is incorrect, for if it were true then the result of Refs. [9-12] would also imply that GUT scale inflation was incompatible with observational data, not just that the observations required the GUT potential to be unreasonably flat.

In this section we analyze the inflation model described by the potential (1.4). We shall make use of a double perturbation expansion, i.e., we will work to first nontrivial order in both spatial irregularity (or the Newton constant) and scalar field nonlinearity (or ϵ). In Sec. 2.1 we study the evolution of the spatially homogeneous background fields. In Sec. 2.2 we use the formalism of Refs. [1,5] to examine spatial irregularities. We catalogue gauge modes in Sec. 2.3 and in Sec. 2.4 we study the asymptotics of our expressions and use the quantum mechanical initial conditions of Ref. [6] to determine the constants of integration that appear in the inflation epoch solutions.

2.1 Spatially Homogeneous Background Fields

The equations of motion for the spatially homogeneous fields are

$$\ddot{\Phi}_b + 3\,\frac{\dot{a}}{a}\,\dot{\Phi}_b + \frac{1}{2}V'(\Phi_b) = 0, \tag{2.4}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{12}\dot{\Phi}_b^2 - \frac{1}{12}V(\Phi_b) = 0, \tag{2.5}$$

where a dot indicates a derivative with respect to time, we have chosen the FLRW synchronous line element

$$ds^2 = dt^2 - a^2(t)(d\vec{x})^2,$$

which describes homogeneous, isotropic, spatially flat cosmological models, $a(t)$ is the scale factor and $\Phi_b(t)$ is the spatially homogeneous background scalar field (the subscript b will be used to denote the spatially homogeneous background part of a field).

Setting $\epsilon = 0$ in eqn. (1.4), the first integral of eqn. (2.4) is

$$\dot{\Phi}_{b0}(t) = \frac{c_0}{{a_0}^3}, \tag{2.6}$$

where the subscript 0 on a field signifies that $\epsilon = 0$ and c_0 is a constant of integration. In terms of an initial value (at $t = t_i$) for the time derivative of the background scalar field, $\dot{\Phi}_{b0}(t_i) = \dot{\Phi}_{b0i}$, we have

$$\dot{\Phi}_{b0}(t) = \dot{\Phi}_{b0i}\left[\frac{a_{0i}}{a_0(t)}\right]^3, \tag{2.7}$$

where a_{0i} is the initial value of the scale factor a_0 at $t = t_i$.

The background scalar field kinetic term in the Hamiltonian density is $\propto a^3(\dot{\Phi}_b)^2$ and the initial conditions of Refs. [17,7,2] for the background fields require that the Hamiltonian density not diverge as $a \to 0$. These initial conditions would require that we choose $c_0 = 0$, however, it is not yet clear whether they are valid for models that have a preinflation epoch.

Given the "decaying" form of eqn. (2.7), one is motivated to neglect the $\dot{\Phi}_{b0}$ term in the $\epsilon = 0$ part of eqn. (2.5). The expanding solution for the scale factor is then

$$a_0(t) \propto e^{H_d(t-t_i)}, \tag{2.8}$$

where $H_d = (\Lambda/6)^{1/2}$; this means that $\dot{\Phi}_{b0}(t) \propto \exp[-3H_d(t-t_i)]$ [from eqn. (2.6)], and so, for reasonable values of c_0, it rapidly vanishes. Hence, at large time, it seems that the difference between the solution $[\dot{\Phi}_{b0}(t)]$ for $c_0 \neq 0$ and that for $c_0 = 0$ should be insignificant. Unfortunately, a slightly more complete analysis, presented in the Appendix, shows that this reasoning is only (somewhat) qualitatively correct. It seems that, provided a certain function of the initial values does not vanish (for details see the Appendix), the qualitative, large time, behaviour of the solutions for the background fields are independent of the specific value chosen for c_0 (i.e., the time dependence of the solutions seems to be generic), however, the dynamics does seem to be quantitatively sensitive to the value of c_0. The case $c_0 \neq 0$ has so far eluded an analytic treatment. Given that the qualitative behaviour of the general case is similar to the $c_0 = 0$ case, we shall use this choice in what follows.

Using this initial condition we find that the spatially homogeneous fields, to order ϵ, are given by

$$\Phi_b(t) = \Phi_{b0i} + \epsilon \left[\bar{c}_0 + \frac{\bar{c}_1}{{a_0}^3} + 2H_d T^3 (t - t_i) \right], \tag{2.9}$$

and

$$a(t) = a_{0i}\, e^{H_d(t-t_i)} \left[1 + \epsilon \left\{ \bar{c}_2 - \frac{1}{8} H_d T^4 (t - t_i) \right\} \right], \tag{2.10}$$

where $\bar{c}_0$, $\bar{c}_1$ and $\bar{c}_2$ are constants of integration, Φ_{b0i} is the value of Φ_{b0} at $t = t_i$, $T = \Phi_{b0i} - \bar{\Phi}$ and H_d is the zeroth order part of the Hubble constant during inflation. If $\bar{c}_0$ and $\bar{c}_1$ are ignored, eqn. (2.9), in the limit $T = 1$, agrees with the solution in the exponential expansion inflation model of Ref. [1] (see eqn. (2.7) there).

We note that in terms of the initial values of the order ϵ fields at $t = t_i$, Φ_{b1i}, $\dot{\Phi}_{b1i}$ and f_i (where $\Phi_b(t) = \Phi_{b0}(t) + \epsilon\, \Phi_{b1}(t)$ and $a(t) = a_0(t)[1 + \epsilon f(t)]$), these equations may be reexpressed as

$$\Phi_{b1}(t) = \Phi_{b1i} + \frac{1}{3H_d} \left(\dot{\Phi}_{b1i} - 2H_d T^3 \right) \left[1 - \left(\frac{a_{0i}}{a_0} \right)^3 \right] + 2H_d T^3 (t - t_i),$$

$$\dot{\Phi}_{b1}(t) = \dot{\Phi}_{b1i} \left(\frac{a_{0i}}{a_0} \right)^3 + 2H_d T^3 \left[1 - \left(\frac{a_{0i}}{a_0} \right)^3 \right],$$

and

$$f(t) = f_i - \frac{1}{8} H_d T^4 (t - t_i).$$

Again, the initial conditions of Refs. [17,7,2] for the background fields would require the choice $\dot{\Phi}_{b1i} = 0$ (if one were allowed to consider the limit $a_0 \ll a_{0i}$); they however do not constrain the values of Φ_{b0i}, Φ_{b1i} and f_i. It might not be unreasonable to absorb Φ_{b1i} in Φ_{b0i} and f_i in a_{0i}, then, with these initial conditions, one would only need an initial condition to determine Φ_{b0i}.

The equations of motion for the spatial irregularities can be integrated without the need for further simplifying assumptions about the initial values of the background fields, so for the present we leave unspecified initial values arbitrary. (It turns out, as might have been anticipated, that the baryon dominated epoch expressions for spatial irregularities are insensitive to the values of $\bar{c}_0$, $\bar{c}_1$ and $\bar{c}_2$.)

2.2 Spatial Irregularities

The synchronous gauge equations governing the evolution of inhomogeneities in a general Einstein-scalar field model have been derived in Sec. 7 of Ref. [3]. These equations are

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + \frac{1}{2}V''(\Phi_b)\phi = \frac{1}{2}\dot{h}\dot{\Phi}_b, \tag{2.11}$$

$$\frac{1}{2}\ddot{h} + \frac{\dot{a}}{a}\dot{h} = \dot{\Phi}_b\dot{\phi} - \frac{1}{4}V'(\Phi_b)\phi, \tag{2.12}$$

$$\dot{h}_{,i} - \dot{h}_{ij,j} = \dot{\Phi}_b\partial_i\phi, \tag{2.13}$$

$$\frac{1}{a^2}(h_{ij,kk} + h_{,ij} - h_{ik,jk} - h_{jk,ik}) - 3\frac{\dot{a}}{a}\dot{h}_{ij} - \frac{\dot{a}}{a}\dot{h}\,\delta_{ij} - \ddot{h}_{ij} = \frac{1}{2}\delta_{ij}V'(\Phi_b)\phi, \tag{2.14}$$

where spatial indices are raised and lowered with the metric $+\delta_{ij}$. To derive these equations we have linearized the metric about a spatially flat, homogeneous and isotropic, FLRW background, with a line element given by

$$ds^2 = dt^2 - a^2(t)[\delta_{ij} - h_{ij}(t,\vec{x})]dx^i dx^j,$$

where h_{ij} are the metric perturbations and h denotes the trace of h_{ij}. We have also linearized the scalar field about a spatially homogeneous background,

$$\Phi(t, \vec{x}) = \Phi_b(t) + \phi(t, \vec{x}).$$

To solve eqns. (2.11) – (2.14) it is convenient to work in spatial momentum space so we Fourier expand the spatial irregularities,

$$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[\phi(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} + \phi^*(t, \vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right],$$

$$h_{ij}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[h_{ij}(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} + h^*_{ij}(t, \vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right],$$

where $\vec{k}$ is the spatial coordinate momentum, $k_i \in [-\infty, \infty]$, $\vec{k}\cdot\vec{x} = \delta_{ij} k_i x_j$ and $\phi^*(t, \vec{k}) = \phi(t, -\vec{k})$. In what follows, we denote $|k_i k_i|^{1/2}$ by k.

We proceed as before, first setting $\epsilon = 0$ in the equations of motion for the perturbations and determining the lowest order solutions for the fields, and then working to order ϵ to derive the first order corrections to these expressions.

The right hand side and the last term on the left hand side of eqn. (2.11) are of order ϵ; ignoring them and solving the resulting equation, we find

$$\phi_0(\vec{k}, t) = c_+ \left(i + \frac{k}{a_0 H_d} \right) e^{ik/(a_0 H_d)} + c_- \left(-i + \frac{k}{a_0 H_d} \right) e^{-ik/(a_0 H_d)}, \tag{2.15}$$

where $c_\pm$ are $\vec{k}$ dependent constants of integration (a part of one linear combination of which corresponds to the adiabatic solution); we shall not have need for the order ϵ correction to this expression. We shall see that this expression does not transform under the remnants of general coordinate invariance in synchronous gauge. The time translation invariance solution (Secs. 3.1 and 3.2 of Ref. [1]), which is gauge-dependent, is not present at lowest order in ϵ.

The right hand side of eqn. (2.12) is also of order ϵ; ignoring it, we find that the lowest order solution is given by

$$h_0(\vec{k}, t) = c_1^{(0)} + \frac{c_5^{(0)}}{{a_0}^2}, \tag{2.16}$$

where $c_1^{(0)}$ and $c_5^{(0)}$ are $\vec{k}$ dependent constants of integration. We shall see that both of these terms are gauge-dependent (the solution proportional to $c_1^{(0)}$ corresponds to the arbitrariness in rescaling a_{0i}). Writing

$$h(\vec{k},t) = h_0(\vec{k},t) + \epsilon\, h_1(\vec{k},t), \tag{2.17}$$

and expanding eqn. (2.12) to order ϵ we have

$$\ddot{h}_1(t) + 2H_d\dot{h}_1(t) = R(t), \tag{2.18}$$

where

$$R(t) = \frac{1}{4}H_d T^4\dot{h}_0 + \left\{-6H_d\frac{\bar{c}_1}{{a_0}^3} + 4H_d T^3\right\}\dot{\phi}_0 + 6{H_d}^2T^3\phi_0, \tag{2.19}$$

and ϕ_0, $\dot{\phi}_0$ and $\dot{h}_0$ may be determined from eqns. (2.15) and (2.16). The solution of eqn. (2.18) is

$$h_1(\vec{k},t) = c_1^{(1)} + \frac{c_5^{(1)}}{{a_0}^2} + \frac{1}{4}\,H_d T^4\,c_5^{(0)}\frac{t}{{a_0}^2} + \left(c_+F_1(w) + c_-F_2(w)\right), \tag{2.20}$$

where $c_1^{(1)}$ and $c_5^{(1)}$ are $\vec{k}$ dependent constants of integration,

$$\begin{aligned} -H_dF_1(w) =& e^{iw}\Big\{\left(12iA_1 - \frac{5}{4}A_2\right) + \left(12A_1 - \frac{i}{4}A_2\right)w - 5iA_1w^2 - A_1w^3\Big\} \\ &+ \frac{3}{4}A_2\ln w + \frac{3}{8}A_2 + \frac{A_2}{8}w^2\ln w - \frac{A_2}{4}w^2\sum_{n=1}^{\infty}\frac{(iw)^n}{n.n!} \\ &- \frac{3}{2}A_2\left\{-iw + \sum_{n=3}^{\infty}\frac{(iw)^n}{n(n-2)n!}\right\}, \end{aligned} \tag{2.21}$$

$F_2(w) = [F_1(w)]^*$, $w = k/(a_0H_d)$ and we have defined

$$A_1 = -\,i\,6\,\bar{c}_1{H_d}^4k^{-3}, \tag{2.22}$$

$$A_2 = i\,4\,H_dT^3. \tag{2.23}$$

We shall see that the first three terms on the right hand side of eqn. (2.20) are gauge-dependent. The $c_1^{(1)}$ term corresponds to the arbitrariness in rescaling a_{0i}, while the $c_\pm$ terms are linear combinations of "isocurvature" and adiabatic perturbations, [18].

Focusing on a plane wave with propagation vector $\vec{k}$ along the x^1 axis and denoting the transverse directions by x^I $(I = 2, 3)$, we find that eqns. (2.13) and (2.14) reduce to

$$\dot{h}_{22}(t) + \dot{h}_{33}(t) = \dot{\Phi}_b \phi(t), \tag{2.24}$$

$$\dot{h}_{I1}(t) = 0, \tag{2.25}$$

$$\ddot{h}_{22}(t) + \ddot{h}_{33}(t) + 3\,\frac{\dot{a}}{a}\,(\dot{h}_{22}(t) + \dot{h}_{33}(t)) = \dot{\Phi}_b \dot{\phi}(t) - \frac{1}{2} V'(\Phi_b)\phi(t), \tag{2.26}$$

$$\frac{k^2}{a^2}(h_{22}(t) + h_{33}(t)) = -2\,\frac{\dot{a}}{a}\,\dot{h}(t) - \dot{\Phi}_b \dot{\phi}(t) - \frac{1}{2} V'(\Phi_b)\phi(t), \tag{2.27}$$

$$\frac{k^2}{a^2} h_+(t) + 3\,\frac{\dot{a}}{a}\,\dot{h}_+(t) + \ddot{h}_+(t) = 0, \tag{2.28}$$

$$\frac{k^2}{a^2} h_\times(t) + 3\,\frac{\dot{a}}{a}\,\dot{h}_\times(t) + \ddot{h}_\times(t) = 0, \tag{2.29}$$

$$\ddot{h}_{I1}(t) + 3\,\frac{\dot{a}}{a}\,\dot{h}_{I1}(t) = 0, \tag{2.30}$$

here $h_+ = h_{22} - h_{33}$ and $h_\times = h_{23} + h_{32}$ are the two physical degrees of freedom of the graviton and we have used $h_{ij} = h_{ji}$ and eqn. (2.12) to simplify some of these equations. Eqn. (2.26) governs the behavior of the perturbations in the curvature of spatial hypersurfaces, eqns. (2.28) and (2.29) are the graviton equations of motion and eqn. (2.30) is the equation of motion for the h_{I1} components of the metric perturbations. Eqns. (2.24), (2.25) and (2.27) are constraint equations.

The solution of eqns. (2.30) and (2.25), to all orders in ϵ, is

$$h_{I1}(\vec{k}, t) = c_{I,1}, \tag{2.31}$$

where $c_{I,1}$ are $\vec{k}$ dependent constants of integration; we shall see that they correspond to gauge-dependent solutions. As discussed in Ref. [1], the form of this equation implies that transverse peculiar velocity perturbations are not present in the large time universe (at least in the linear perturbation approximation).

The solutions of the graviton equations of motion, eqns. (2.28) and (2.29), to lowest order in ϵ, are

$$h_{i(0)}(\vec{k},t) = c_{i1}\left(i + \frac{k}{a_0 H_d}\right) e^{ik/(a_0 H_d)} + c_{i2}\left(-i + \frac{k}{a_0 H_d}\right) e^{-ik/(a_0 H_d)}, \tag{2.32}$$

where $i = +, \times$ and c_{i1} and c_{i2} are $\vec{k}$ dependent constants of integration (and correspond to gauge-invariant solutions); we shall not have need for the order ϵ corrections to these solutions.

The right hand side of eqn. (2.26) is of order ϵ; ignoring it, we find that the lowest order solution is given by

$$h_{22(0)}(\vec{k},t) + h_{33(0)}(\vec{k},t) = c_7^{(0)} + \frac{c_8^{(0)}}{{a_0}^3}, \tag{2.33}$$

where $c_7^{(0)}$ and $c_8^{(0)}$ are $\vec{k}$ dependent constants of integration. We shall see that $c_7^{(0)}$ is related to $c_5^{(0)}$ [eqn. (2.16)] and that it corresponds to a gauge-dependent solution. We shall also find that $c_8^{(0)}$ must vanish. Writing

$$h_{22} + h_{33} = h_{22(0)} + h_{33(0)} + \epsilon\,(h_{22(1)} + h_{33(1)}), \tag{2.34}$$

we find that the order ϵ part of eqn. (2.26) is

$$\ddot{h}_{22(1)} + \ddot{h}_{33(1)} + 3H_d\,(\dot{h}_{22(1)} + \dot{h}_{33(1)}) = S(t), \tag{2.35}$$

where we have defined

$$S(t) = -\frac{9}{8}\,{H_d}^2 T^4\,\frac{c_8^{(0)}}{{a_0}^3} + \left\{-3H_d\frac{\bar{c}_1}{{a_0}^3} + 2H_d T^3\right\}\dot{\phi}_0 + 6{H_d}^2 T^3 \phi_0, \tag{2.36}$$

and ϕ_0 and $\dot{\phi}_0$ may be determined from eqn.(2.15). The solution of eqn. (2.35) is

$$h_{22(1)}(t) + h_{33(1)}(t) = c_7^{(1)} + \frac{c_8^{(1)}}{{a_0}^3} + \frac{3}{8}\,H_d T^4\,c_8^{(0)}\frac{t}{{a_0}^3} + \left(c_+\widetilde{F}_1(w) + c_-\widetilde{F}_2(w)\right), \tag{2.37}$$

where $c_7^{(1)}$ and $c_8^{(1)}$ are $\vec{k}$ dependent constants of integration,

$$-H_d\widetilde{F}_1(w) = e^{iw}\left[\left(4iA_1 - \frac{2}{3}A_2\right) + \left(4A_1 - \frac{i}{12}A_2\right)w + \left(-2iA_1 + \frac{1}{12}A_2\right)w^2 - \frac{A_1}{2}w^3\right]$$

$$+\frac{A_2}{2}\ln w+\frac{A_2}{6}-\frac{i}{12}A_2w^3\sum_{n=1}^{\infty}\frac{(iw)^n}{n.n!}$$
$$-\frac{3}{2}A_2\Big[-\frac{i}{2}w+\frac{1}{4}w^2+\sum_{n=4}^{\infty}\frac{(iw)^n}{n(n-3)n!}\Big], \tag{2.38}$$

$\widetilde{F}_2(w)=[\widetilde{F}_1(w)]^*$ and w, A_1 and A_2 are defined above. We shall find that part of $c_7^{(1)}$ corresponds to a gauge-dependent solution while part of it describes a gauge-invariant term and that it is related to $c_5^{(0)}$, $c_5^{(1)}$ and $c_\pm$, while $c_8^{(1)}$ must be related to $c_\pm$ and hence characterizes a gauge-invariant solution. The $c_\pm$ terms correspond to adiabatic fluctuations [we shall see when we analyze the long wavelength asymptotics of eqn. (2.37), actually eqn. (2.44) below, that one linear combination of the $c_\pm$ terms present in the scalar field solution, eqn (2.15), is subdominant in the large scale asymptotic expansion of the solution for $h_{22}+h_{33}$ — this is the "isocurvature", [18], or longitudinal peculiar velocity mode].

From eqn. (2.33) we see that to satisfy eqn. (2.24) to order ϵ^0 we must require

$$c_8^{(0)}=0, \tag{2.39}$$

while from eqns. (2.9), (2.15), (2.34), (2.37) and (2.38) we find that the order ϵ part of eqn. (2.24) requires that

$$c_8^{(1)}=-\frac{13}{18}(c_++c_-)\left(\frac{kT}{H_d}\right)^3. \tag{2.40}$$

From the lowest order part of the constraint equation (2.27) we find, using eqns. (2.16) and (2.33), that

$$c_7^{(0)}=4\frac{{H_d}^2}{k^2}c_5^{(0)}, \tag{2.41}$$

while the order ϵ part of this equation results in

$$c_7^{(1)}=4\frac{{H_d}^2}{k^2}\left[c_5^{(0)}\left\{2\bar{c}_2+\frac{1}{4}T^4(H_dt_i-1)\right\}+c_5^{(1)}\right]-12iT^3(c_+-c_-). \tag{2.42}$$

To recapitulate, the solutions of the equations of motion for spatial irregularities, eqns. (2.11) – (2.14), are given by eqns. (2.15), (2.16), (2.20), (2.31) – (2.33) and (2.37), where

$c_7^{(0)}$ is related to $c_5^{(0)}$ [eqn. (2.41)], $c_8^{(0)}$ vanishes [eqn. (2.39)], $c_7^{(1)}$ is determined in terms of $c_5^{(0)}$, $c_5^{(1)}$ and $c_\pm$ [eqn. (2.42)] and $c_8^{(1)}$ is related to $c_\pm$ [eqn. (2.40)]. This leaves a number of $\vec{k}$ dependent constants of integration (some of which describe gauge-dependent solutions) that must be determined from initial conditions. These are: $c_1^{(0)}$ and $c_1^{(1)}$ which describe the gauge-dependent solution corresponding to the rescaling of a_{0i}; $c_5^{(0)}$ and $c_5^{(1)}$ which may be thought of as characterizing the gauge-dependent time translation invariance solution; $c_\pm$ which describe gauge-invariant adiabatic and "isocurvature" fluctuations; $c_{I,1}$ which describe the gauge-dependent h_{I1} components of the metric perturbations and $c_{+1}, c_{+2}, c_{\times 1}$ and $c_{\times 2}$ which are a gauge-invariant characterization of gravitons.

We now tabulate various combinations of the above expressions for the "fundamental" fields that we shall have need for.

The metric perturbations $h_{22} + h_{33}$ measure perturbations in the curvature of spatial hypersurfaces. The Ricci tensor of the three dimensional spatial hypersurface is given by

$$ {}^{(3)}R_{ij} = \frac{1}{2}(h_{ij,kk} + h_{,ij} - h_{ik,kj} - h_{jk,ki}), \tag{2.43} $$

so we have for the induced spatial scalar curvature,

$$ {}^{(3)}R = \frac{k^2}{a^2}(h_{22}(x) + h_{33}(x)) = {}^{(3)}R_{(0)} + \epsilon\ {}^{(3)}R_{(1)} + \cdots, \tag{2.44} $$

where, from eqns. (2.33), (2.39) and (2.41),

$$ {}^{(3)}R_{(0)} = 4\frac{{H_d}^2}{{a_0}^2} c_5^{(0)}, \tag{2.45} $$

which is gauge-dependent. From the above equations as well as eqns. (2.10) and (2.37), we find,

$$ {}^{(3)}R_{(1)} = \frac{{H_d}^2}{{a_0}^2}\left[T^4 c_5^{(0)}(H_d t - 1) + 4c_5^{(1)}\right] + H_d\left(c_+ \bar{F}_1(w) + c_- \bar{F}_2(w)\right), \tag{2.46} $$

where we have defined

$$ \bar{F}_1(w) = e^{iw}\left[\left(-4iA_1 + \frac{2}{3}A_2\right)w^2 + \left(-4A_1 + \frac{i}{12}A_2\right)w^3 + \left(2iA_1 - \frac{1}{12}A_2\right)w^4 + \frac{1}{2}A_1 w^5\right] $$

$$
-\frac{1}{2}A_2w^2\ln w-\frac{19}{6}A_2w^2+i\frac{13}{72}A_2w^5+\frac{i}{12}A_2w^5\sum_{n=1}^{\infty}\frac{(iw)^n}{n.n!}
$$

$$
+\frac{3}{2}A_2\left[-\frac{i}{2}w^3+\frac{1}{4}w^4+w^2\sum_{n=4}^{\infty}\frac{(iw)^n}{n(n-3)n!}\right], \tag{2.47}
$$

$\bar{F}_2(w) = [\bar{F}_1(w)]^*$ and w, A_1 and A_2 have been defined above. We shall see that the first term in eqn. (2.46) (the term with the constants $c_5^{(0)}$ and $c_5^{(1)}$) is a gauge-dependent solution, while the leading (in the large-scale asymptotic expansion) linear combination of terms proportional to $c_\pm$ is, by definition, the adiabatic perturbation.

An alternate measure of energy density perturbations is the linearized stress-energy tensor. From eqns. (2.3) and (7.10) of Ref. [3] we have that the fractional perturbation in energy density corresponding to the scalar field perturbation ϕ is

$$
\delta_\Phi = \frac{\delta\rho_\Phi}{\rho_{b\Phi}} = 2\frac{\left\{\dot{\Phi}_b\dot{\phi}+V'(\Phi_b)\phi/2\right\}}{\left\{(\dot{\Phi}_b)^2+V(\Phi_b)\right\}}; \tag{2.48}
$$

we note that the scalar field background energy density and pressure are

$$
\rho_{b\Phi} = \frac{m_p^2}{32\pi}\left[(\dot{\Phi}_b)^2+V(\Phi_b)\right], \tag{2.49}
$$

$$
p_{b\Phi} = \frac{m_p^2}{32\pi}\left[(\dot{\Phi}_b)^2-V(\Phi_b)\right]. \tag{2.50}
$$

Evaluating this expression to lowest order in ϵ we find the gauge-invariant expression

$$
\delta_\Phi(\vec{k},t) = \frac{\epsilon}{H_d}\left(c_+G_1(w)+c_-G_2(w)\right), \tag{2.51}
$$

where we have defined

$$
G_1(w) = e^{iw}\left[-\frac{A_1}{12}w^5-\frac{A_2}{12}w^2+\frac{i}{4}A_2w-\frac{A_2}{4}\right], \tag{2.52}
$$

and $G_2(w) = [G_1(w)]^*$.

We shall also have need for (eqn. (7.11) of Ref. [3])

$$(\rho_{b\Phi} + p_{b\Phi})\, v^i_\Phi = -\,\frac{m_p^2}{16\pi}\,\frac{\dot{\Phi}_b}{a}\,\partial_i\phi, \tag{2.53}$$

where we may call v^i_Φ the peculiar velocity corresponding to the scalar field perturbation ϕ, as well as (eqn. (4.20) of Ref. [1])

$$c_{s\Phi}^2\,\rho_{b\Phi}\,\delta_\Phi = \frac{m_p^2}{16\pi}\left[\dot{\Phi}_b\dot{\phi} - \frac{1}{2}V'(\Phi_b)\phi\right], \tag{2.54}$$

where we may call $c_{s\Phi}$ the effective speed of propagation of "acoustic" disturbances in the scalar field model (no physical significance is to be attributed to this name — this is not a physical speed).

We find, from eqn. (2.53), that, during inflation, transverse peculiar velocity perturbations vanish,

$$v^I_\Phi(\vec{k}, t) = 0, \tag{2.55}$$

as well as, to lowest order in ϵ,

$$(\rho_{b\Phi} + p_{b\Phi})\, v^1_\Phi(\vec{k}, t) = -\,\frac{m_p^2}{32\pi}\,\epsilon H_d\left(c_+\widetilde{G}_1(w) + c_-\widetilde{G}_2(w)\right), \tag{2.56}$$

where we have defined

$$\widetilde{G}_1(w) = e^{iw}\left[A_1w^5 + iA_1w^4 + A_2w^2 + iA_2w\right] \tag{2.57}$$

and $\widetilde{G}_2(w) = [\widetilde{G}_1(w)]^*$. Eqns. (2.55) and (2.56) are gauge-invariant. From eqn. (2.54) we find, to lowest order in ϵ,

$$c_{s\Phi}^2\,\rho_{b\Phi}\,\delta_\Phi = -\,\frac{m_p^2}{32\pi}\,\epsilon H_d\left(c_+\bar{G}_1(w) + c_-\bar{G}_2(w)\right) \tag{2.58}$$

(which is gauge-invariant),where

$$\bar{G}_1(w) = e^{iw}\left[A_1w^5 + A_2w^2 + 3iA_2w - 3A_2\right] \tag{2.59}$$

and $\bar{G}_2(w) = [\bar{G}_1(w)]^*$.

2.3 Remnants of General Coordinate Invariance

Consistent with the choice of synchronous gauge, one may still perform time-independent gauge transformations (i.e., the corresponding gauge parameters are time-independent) on the perturbations without altering the physics. To zeroth order in ϵ the metric transformations are

$$
\begin{aligned}
\delta h_{11(0)}(k,t) &= -\frac{1}{H_d}\left(\frac{k}{a_0}\right)^2 f^0_{(0)}(k) - 2ik\,\omega_{1(0)}(k) - 2H_d f^0_{(0)}(k), \\
\delta h_{1I(0)}(k,t) &= -ik\,\omega_{I(0)}(k), \\
\delta h_{IJ(0)}(k,t) &= -2H_d f^0_{(0)}(k)\,\delta_{IJ}, \\
\delta h_{+(0)}(k,t) &= 0 = \delta h_{\times(0)}(k,t), \\
\delta h_0(k,t) &= -\frac{1}{H_d}\left(\frac{k}{a_0}\right)^2 f^0_{(0)}(k) - 2ik\,\omega_{1(0)}(k) - 6H_d f^0_{(0)}(k),
\end{aligned} \tag{2.60}
$$

where the zeroth order gauge parameters $f^0_{(0)}(k)$ and $\omega_{i(0)}(k)$ do not depend on t. The transformations of the first order (in ϵ) part of the metric perturbations are

$$
\begin{aligned}
\delta h_{11(1)}(k,t) &= -\frac{1}{H_d}\left(\frac{k}{a_0}\right)^2 f^0_{(1)}(k) - 2ik\,\omega_{1(1)}(k) - 2H_d f^0_{(1)}(k) \\
&\quad + 2k^2 f^0_{(0)}(k)\,W(t) + \frac{1}{4}H_d T^4 f^0_{(0)}(k), \\
\delta h_{1I(1)}(k,t) &= -ik\,\omega_{I(1)}(k), \\
\delta h_{IJ(1)}(k,t) &= -2H_d f^0_{(1)}(k)\,\delta_{IJ} + \frac{1}{4}H_d T^4 f^0_{(0)}(k)\,\delta_{IJ}, \\
\delta h_{+(1)}(k,t) &= 0 = \delta h_{\times(1)}(k,t), \\
\delta h_1(k,t) &= -\frac{1}{H_d}\left(\frac{k}{a_0}\right)^2 f^0_{(1)}(k) - 2ik\,\omega_{1(1)}(k) - 6H_d f^0_{(1)}(k) \\
&\quad + 2k^2 f^0_{(0)}(k)\,W(t) + \frac{3}{4}H_d T^4 f^0_{(0)}(k),
\end{aligned} \tag{2.61}
$$

where we have defined

$$
W(t) = \frac{1}{H_d {a_0}^2}\left[\bar{c}_2 - \frac{1}{16}T^4 - \frac{1}{8}T^4 \ln\left(\frac{a_0}{a_{0i}}\right)\right], \tag{2.62}
$$

and the first order gauge parameters $f^0_{(1)}(k)$ and $\omega_{i(1)}(k)$ are t and ϵ independent [where $f^0(k) = f^0_{(0)}(k) + \epsilon\, f^0_{(1)}(k) + \cdots$ and similarly for $\omega_i(k)$]. It is straightforward to verify that the choices

$$f^0_{(0)}(k) = -\frac{H_d}{k^2}\, c_5^{(0)}\,, \tag{2.63}$$

$$\omega_{1(0)}(k) = \frac{i}{2k}\, c_1^{(0)} - 3i\, \frac{{H_d}^2}{k^3}\, c_5^{(0)}\,, \tag{2.64}$$

and

$$\omega_{I(0)}(k) = \frac{i}{k}\, c_{I,1(0)}\,, \tag{2.65}$$

generate the two terms in eqn. (2.16), the order ϵ^0 part of the solution (2.31), the first term in eqn. (2.33) [here we use eqns. (2.39) and (2.41)] and the solution in eqn. (2.45). Similarly, it may be shown that the above choices along with

$$f^0_{(1)}(k) = \frac{H_d}{k^2}\left[c_5^{(0)}\left\{-2\bar{c}_2 + \frac{1}{8}T^4(1-2H_d t_i)\right\} - c_5^{(1)}\right], \tag{2.66}$$

$$\omega_{1(1)}(k) = \frac{i}{2k}\, c_1^{(1)} + 3i\, \frac{{H_d}^2}{k^3}\left[2c_5^{(0)}\left\{-\bar{c}_2 + \frac{1}{8}T^4(1-H_d t_i)\right\} - c_5^{(1)}\right], \tag{2.67}$$

and

$$\omega_{I(1)}(k) = \frac{i}{k} c_{I,1(1)}\,, \tag{2.68}$$

generate the first three terms on the right hand side of eqn. (2.20), the order ϵ part of the solution in eqn. (2.31), the gauge-dependent part of the $c_7^{(1)}$ term in eqn. (2.37) [where we use eqn. (2.42)] and the first term on the right hand side of eqn. (2.46). The two transverse degrees of freedom of the graviton are gauge-invariant, to all orders in ϵ.

We now determine the transformation properties of the matter part of the model. From the definition and transformation properties of the stress-energy tensor, we find

$$\delta\phi(k,t) = f^0(k)\, \dot{\Phi}_b, \tag{2.69}$$

whose leading term is of order ϵ. So, as claimed, eqn. (2.15) is gauge-invariant. We also have

$$\delta\left[\delta_\Phi(k,t)\right] = f^0(k)\,\frac{\dot\rho_{b\Phi}}{\rho_{b\Phi}}, \tag{2.70}$$

$$\delta v^I_\Phi(k,t) = 0, \tag{2.71}$$

$$\delta\left[\left(\rho_{b\Phi} + p_{b\Phi}\right) v^1_\Phi(k,t)\right] = -\,\frac{m_p^2}{16\pi}\,\frac{ik}{a}\,\dot\Phi_b{}^2\, f^0(k), \tag{2.72}$$

and

$$\delta\left[c^2_{s\Phi}\,\rho_{b\Phi}\,\delta_\Phi(k,t)\right] = f^0(k)\,\dot p_{b\Phi}. \tag{2.73}$$

It is straightforward to verify that the leading terms on the right hand sides of eqns. (2.70), (2.72) and (2.73) are of order ϵ^2. As a result eqns. (2.51), (2.55), (2.56) and (2.58) are gauge-invariant. As noted in Ref. [1] [see the discussion following eqn. (5.18) there], the gauge-dependent time translation solution does not appear in the lowest order (in ϵ) expressions for the matter variables, in the exponential expansion inflation limit.

To summarize, of the many k dependent constants of integration, only six correspond to gauge-invariant solutions: $c_\pm$, which characterize matter perturbations and c_{+1}, c_{+2}, $c_{\times 1}$ and $c_{\times 2}$, which correspond to gravitational wave perturbations. These constants of integration are determined from the quantum mechanical initial conditions of Refs. [6,7,1]. Those constants of integration which correspond to gauge-dependent solutions are arbitrary, i.e., there are no initial conditions which determine them, [1].

The induced spatial curvature is a measure of the magnitude of adiabatic energy density fluctuations, however, the expression in eqn. (2.44) is not gauge-invariant. A gauge-invariant characterization of energy density fluctuations, [7,5], is

$$\Delta\rho_\Phi = a^3\left[{}^{(3)}R + \frac{\dot a}{a}\frac{\partial}{\partial t}\left\{\frac{3}{\nabla^2}h_{ij,ij} - h\right\}\right], \tag{2.74}$$

where $\nabla^2 = \delta_{ij}\partial_i\partial_j$. For our purposes, it is more convenient to use (eqn. (4.25) of Ref. [1])

$$\Delta_\Phi = -\frac{m_p^2}{16\pi}\,\frac{\Delta\rho_\Phi}{a^3\rho_{b\Phi}}\,. \tag{2.75}$$

We shall also have need for another gauge-invariant variable (eqn. (4.24) of Ref. [1]),

$$A_\Phi = \delta_\Phi - \frac{3}{4}\,\frac{(\rho_{b\Phi} + p_{b\Phi})}{\rho_{b\Phi}}\,(h_{22} + h_{33})\,. \tag{2.76}$$

These variables obey relatively simple gauge-invariant joining conditions at the transitions. Although, in the exponential expansion inflation model, to the order (in ϵ) to which we work, the more standard variables (for instance, δ_Φ) are gauge-invariant, the variables in eqns. (2.75) and (2.76) (along with the gauge-invariant joining conditions of Ref. [4]) will provide a check on our algebraic manipulations at the transitions.

From eqns. (2.74) and (2.75) we find

$$\Delta_\Phi = \frac{\epsilon}{H_d}\,\left(c_+\hat{F}_1(w) + c_-\hat{F}_2(w)\right), \tag{2.77}$$

where

$$\hat{F}_1(w) = e^{iw}\left[-\frac{A_1}{12}w^5 - \frac{i}{4}A_1 w^4 + \frac{A_1}{4}w^3 - \frac{A_2}{12}w^2\right], \tag{2.78}$$

and $\hat{F}_2(w) = [\hat{F}_1(w)]^*$. The prefactor in the second term on the right hand side of eqn. (2.76) is of order ϵ^2, so

$$A_\Phi = \delta_\Phi, \tag{2.79}$$

where δ_Φ is given in eqns. (2.51) and (2.52).

2.4 Asymptotics and Initial Conditions

The constants of integration, $c_\pm, c_{+1}, c_{+2}, c_{\times 1}$ and $c_{\times 2}$ (which correspond to gauge-invariant solutions) are determined by requiring that the small-scale (i.e., $k \gg \dot{a}$) part of the inflation epoch solutions reduce to the usual quantum mechanical "vacuum state" form, [6,7,1]. [We are justified in using initial conditions for these perturbations since they are the perturbations of interest in the large time universe, the preinflation perturbations (the form of which are also governed by these quantum mechanical initial conditions) are not of large time interest.] In this section we also derive large-scale asymptotic forms of some of these solutions.

We note that the variable w is related to conformal time τ through

$$w = \frac{k}{a_0 H_d} = -k \int^{t} \frac{dt'}{a_0(t')} = -k\tau, \tag{2.80}$$

which is valid to lowest order in ϵ.

Following Sec. 3.3 of Ref. [1], if we define the time-independent constants $\widetilde{c}_\pm$ by

$$c_\pm = \widetilde{c}_\pm \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{1}{\sqrt{2}} H_d\, k^{-3/2}\,, \tag{2.81}$$

we find in the short wavelength (early time) limit that the order ϵ^0 part of the scalar field perturbation, eqn. (2.15), reduces to

$$\phi_0(\vec{k},t) = \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{1}{\sqrt{2ka_0^2}} \left[\widetilde{c}_+ \exp\left(-ik\int^{t} \frac{dt'}{a_0(t')}\right) + \widetilde{c}_- \exp\left(+ik\int^{t} \frac{dt'}{a_0(t')}\right)\right]; \tag{2.82}$$

relativistic covariance, see Sec. 3.3 of Ref. [1], requires that we choose

$$\widetilde{c}_+ = 1 \quad \text{and} \quad \widetilde{c}_- = 0. \tag{2.83}$$

For the gravitational wave perturbations, it is convenient to define the time-independent constants $\widetilde{c}_{i1}, \widetilde{c}_{i2}$ (where $i = +, \times$) through

$$c_{i\binom{1}{2}} = \widetilde{c}_{i\binom{1}{2}} \left(\frac{16\pi}{m_p^2}\right)^{1/2} \sqrt{2}\, H_d\, k^{-3/2}\,; \tag{2.84}$$

we then find on small scales that the order ϵ^0 expressions for the graviton, eqns. (2.32), become

$$h_{i(0)}(\vec{k},t) = \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{2}{\sqrt{2ka_0^2}} \left[\widetilde{c}_{i1} \exp\left(-ik\int^{t} \frac{dt'}{a_0(t')}\right) + \widetilde{c}_{i2} \exp\left(+ik\int^{t} \frac{dt'}{a_0(t')}\right)\right]; \tag{2.85}$$

the appropriate gravitational wave "vacuum state" initial conditions are $\widetilde{c}_{i1} = 1$ and $\widetilde{c}_{i2} = 0$. The extra factor of 2 in the numerator of the prefactor of eqn. (2.85) [relative to that of eqn.

(2.82)] is a consequence of the extra factor of 1/4 in the graviton kinetic term of the action (2.1), see Sec. 3.3 of Ref. [1]. Using these results, it may be shown that the expressions for gravitational wave perturbations, eqns. (2.32), agree with the de Sitter inflation limit of the expressions given in Ref. [1] (see eqns. (3.26), (3.27) and (3.44) there); we shall record some of the intermediate results for these perturbations in what follows since we did not do so in Ref. [1].

Scalar field fluctuations outside the Hubble radius (i.e., in the long wavelength or large time limit, $k \ll \dot{a}$) are asymptotically described by

$$\phi_0(\vec{k}, t) = c_{A1}\, k^{-3/2} + c_{A2}\, k^{1/2}(a_0)^{-2} + c_I\, k^{3/2}(a_0)^{-3} + \cdots, \tag{2.86}$$

where the coefficients are given by

$$\begin{aligned} c_{A1} =& \frac{1}{\sqrt{2}} \left(\frac{16\pi}{m_p^2}\right)^{1/2} H_d\, i\, (\widetilde{c}_+ - \widetilde{c}_-), \\ c_{A2} =& \frac{1}{2\sqrt{2}} \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{1}{H_d}\, i\, (\widetilde{c}_+ - \widetilde{c}_-), \\ c_I =& -\frac{1}{3\sqrt{2}} \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{1}{H_d{}^2} (\widetilde{c}_+ + \widetilde{c}_-), \end{aligned} \tag{2.87}$$

It is pleasing to note that the time dependence and spatial momentum dependence of the second and third terms in eqn. (2.86) agree with that of the terms in eqn. (3.46) of Ref. [1] (in the exponential expansion inflation limit). [These terms are, in fact, identical (as they must be). The first term in (2.86) was not recorded in eqn. (3.46) of Ref. [1] — we shall show that the scalar field two-point functions for these models are identical.] We note that the scalar field perturbation has no growing mode. The terms proportional to c_{A1} and c_{A2} may be called adiabatic solutions while the term proportional to c_I may be called the "isocurvature" solution — it does not contribute to spatial curvature perturbations on large scales (although it does perturb spatial curvature on small scales), [18].

On large scales, the expressions for the graviton, eqns. (2.32), have a similar asymptotic expansion,

$$h_{i(0)}(\vec{k},t) = c_i^1\, k^{-3/2} + c_i^2\, k^{1/2}(a_0)^{-2} + c_i^3\, k^{3/2}(a_0)^{-3} + \cdots, \tag{2.88}$$

where we have defined

$$\begin{aligned} c_i^1 =& \sqrt{2}\left(\frac{16\pi}{m_p^2}\right)^{1/2} H_d\, i\, (\widetilde{c}_{i1} - \widetilde{c}_{i2}), \\ c_i^2 =& \frac{1}{\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{1}{H_d}\, i\, (\widetilde{c}_{i1} - \widetilde{c}_{i2}), \\ c_i^3 =& -\frac{\sqrt{2}}{3}\left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{1}{{H_d}^2}\, (\widetilde{c}_{i1} + \widetilde{c}_{i2}), \end{aligned} \tag{2.89}$$

The time and spatial momentum dependence of the first and third terms in eqn. (2.88) agree with that of the leading terms in eqn. (3.47) of Ref. [1]. We show below that the gravitational wave two-point functions of these models are identical.

We note the obvious fact that the order ϵ^0 scalar field and gravitational wave perturbations do not sense the presence of the order ϵ term in the potential, eqn. (1.4) (i.e., we can set $T = 0$ and these expressions will not change).

On large scales we find that the curvature of spatial hypersurfaces, eqns. (2.44) – (2.47), is given by [setting $c_5^{(0)} = 0 = c_5^{(1)}$],

$$\begin{aligned} {}^{(3)}R(\vec{k},t) = \frac{\epsilon}{(a_0)^2}\Bigg[& \hat{c}_{1A}\, k^{1/2} + \hat{c}_1\, k^{-5/2} + \hat{c}_2\, k^{1/2} \ln\left(\frac{k}{a_0 H_d}\right) \\ & + \hat{c}_3\, k^{5/2}\,(a_0)^{-2} + \hat{c}_{4I}\, k^{7/2}\,(a_0)^{-3} + \hat{c}_4\, k^{1/2}\,(a_0)^{-3} + \cdots \Bigg], \end{aligned} \tag{2.90}$$

where the coefficients are

$$\hat{c}_{1A} = -\frac{5}{2\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2} A_2(\widetilde{c}_+ - \widetilde{c}_-), \tag{2.91}$$

$$\hat{c}_1 = -\frac{4}{\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2} i\, A_1\, k^3(\widetilde{c}_+ + \widetilde{c}_-), \tag{2.92}$$

$$\hat{c}_2 = -\frac{1}{2\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2} A_2(\widetilde{c}_+ - \widetilde{c}_-), \tag{2.93}$$

$$\hat{c}_3 = -\frac{1}{8\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{A_2}{{H_d}^2}(\widetilde{c}_+ - \widetilde{c}_-), \tag{2.94}$$

$$\hat{c}_{4I} = -\frac{1}{18\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{iA_2}{{H_d}^3}(\widetilde{c}_+ + \widetilde{c}_-), \tag{2.95}$$

$$\hat{c}_4 = -\frac{1}{6\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{A_1 k^3}{{H_d}^3}(\widetilde{c}_+ - \widetilde{c}_-). \tag{2.96}$$

Before attempting to describe the physical significance of these formulae we note that even though the lowest order scalar field perturbation is not sensitive to the order ϵ term in the potential, eqn. (1.4), the lowest order nontrivial term in the curvature of spatial hypersurfaces does sense the presence of the order ϵ term in the potential [see eqn. (2.23)]. Ignoring terms proportional to $\bar{c}_1$ [see eqns. (2.9) and (2.22)] and focussing on those terms sensitive to the order ϵ part of the potential, the first (dominant) time-independent term in the square parentheses in eqn. (2.90) [the term proportional to $\hat{c}_{1A}$, eqn. (2.91)] may be called the adiabatic solution. The terms proportional to $\hat{c}_2$ and $\hat{c}_3$ are less significant adiabatic solutions, while the subdominant $\hat{c}_{4I}$ term is what we have been calling the "isocurvature" solution. We note that the time and spatial momentum dependence of the $\hat{c}_{1A}$, $\hat{c}_3$ and $\hat{c}_{4I}$ terms agree with the leading terms in eqn. (3.49) of Ref. [1] (in the de Sitter inflation limit). The terms proportional to $\bar{c}_1$ (or A_1), i.e., the $\hat{c}_1$ and $\hat{c}_4$ terms in eqn. (2.90) have the same time dependence (but different dependence on spatial momentum) as other terms in this equation.

For large scale fluctuations, we find that δ_Φ, eqn. (2.51), is given by

$$\delta_\Phi(\vec{k},t) = \epsilon\left[\check{c}_{A1}\,k^{-3/2} + \check{c}_{A2}\,k^{1/2}\,(a_0)^{-2} + \check{c}_I\,k^{3/2}\,(a_0)^{-3} + \cdots\right], \tag{2.97}$$

where we have defined

$$\check{c}_{A1} = -\frac{1}{4\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2} A_2\,(\widetilde{c}_+ - \widetilde{c}_-), \tag{2.98}$$

$$\check{c}_{A2} = -\frac{5}{24\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{A_2}{{H_d}^2}\,(\widetilde{c}_+ - \widetilde{c}_-), \tag{2.99}$$

$$\breve{c}_I = -\frac{1}{6\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{iA_2}{{H_d}^3}\,(\widetilde{c}_+ + \widetilde{c}_-). \tag{2.100}$$

We note that the leading terms in this asymptotic expansion are insensitive to the value of $\bar{c}_1$ [eqn. (2.9)]. There are no growing modes; the first two terms (proportional to $\breve{c}_{A1}$ and $\breve{c}_{A2}$) are the dominant and subdominant adiabatic solutions, while the term proportional to $\breve{c}_I$ is the "isocurvature" solution. The time and spatial momentum dependence of the second and third terms in eqn. (2.97) agrees with the leading, exponential expansion inflation limit, behaviour of eqn. (3.51) of Ref. [1]; the time and spatial momentum dependence of the terms in eqn. (2.97) also agree with those in eqn. (2.86).

As noted in Refs. [3,1], during inflation, scalar field perturbations, eqn. (2.86), and fractional energy density perturbations, eqn. (2.97), do not grow, even in the presence of nonzero space curvature fluctuations, eqn. (2.90). This is to be contrasted with the behavior of the adiabatic solution in ideal fluid dominated cosmologies where a nonzero space curvature irregularity is always accompanied by a growing fractional energy density irregularity.

On large scales, we find that the gauge-invariant measure of energy density perturbations, Δ_Φ, eqn. (2.77), becomes

$$\begin{aligned}\Delta_\Phi(\vec{k},t) = \epsilon\Big[&\breve{c}_{1A}\,k^{1/2}\,(a_0)^{-2} + \breve{c}_{2I}\,k^{3/2}\,(a_0)^{-3}\\ &+ \breve{c}_2\,k^{-3/2}\,(a_0)^{-3} + \breve{c}_{3A}\,k^{5/2}\,(a_0)^{-4} + \cdots\Big],\end{aligned} \tag{2.101}$$

where we have defined

$$\breve{c}_{1A} = -\frac{1}{12\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{A_2}{{H_d}^2}(\widetilde{c}_+ - \widetilde{c}_-), \tag{2.102}$$

$$\breve{c}_{2I} = -\frac{1}{12\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{iA_2}{{H_d}^3}(\widetilde{c}_+ + \widetilde{c}_-), \tag{2.103}$$

$$\breve{c}_2 = \frac{1}{4\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{A_1k^3}{{H_d}^3}(\widetilde{c}_+ - \widetilde{c}_-), \tag{2.104}$$

$$\breve{c}_{3A} = \frac{1}{24\sqrt{2}}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{A_2}{{H_d}^4}(\widetilde{c}_+ - \widetilde{c}_-). \tag{2.105}$$

Focussing on the $\bar{c}_1$ independent terms, the terms in eqn. (2.101) proportional to $\breve{c}_{1A}$ and $\breve{c}_{3A}$ are the dominant and the subdominant leading adiabatic solutions and the $\breve{c}_{2I}$ term is the leading "isocurvature" solution. We emphasize that Δ_Φ does evolve on scales larger than the Hubble scale.

In concluding this section we tabulate various momentum space two-point functions. The scalar field perturbation two-point function is

$$\langle \phi_0(k)\, \phi_0^*(k) \rangle = \frac{8\pi}{m_p^2} \frac{{H_d}^2}{k^3} \left[1 + \left(\frac{k}{a_0 H_d} \right)^2 \right], \tag{2.106}$$

which agrees with eqn. (5.18) of Ref. [6]. The gravitational wave perturbation two-point function is

$$\left\langle h_{+(0)}(k)\, h_{+(0)}^*(k) \right\rangle = \frac{32\pi}{m_p^2} \frac{{H_d}^2}{k^3} \left[1 + \left(\frac{k}{a_0 H_d} \right)^2 \right]; \tag{2.107}$$

this agrees with eqn. (3.58) of Ref. [1]. The fractional energy density perturbation two-point function is given by

$$\begin{aligned} \langle \delta_\Phi(k)\, \delta_\Phi^*(k) \rangle = \frac{16\pi}{m_p^2}\, \epsilon^2 \bigg[& \frac{1}{2}\, {H_d}^2\, T^6\, k^{-3} + \frac{5}{6}\, T^6\, k^{-1}\, (a_0)^{-2} \\ & + \frac{1}{18}\, {H_d}^{-2}\, T^6\, k\, (a_0)^{-4} - \frac{1}{2}\, \bar{c}_1\, T^3\, k^{-1}\, (a_0)^{-5} \\ & - \frac{1}{6}\, \bar{c}_1\, {H_d}^{-2}\, T^3\, k\, (a_0)^{-7} + \frac{1}{8}\, {\bar{c}_1}^2\, {H_d}^{-2}\, k\, (a_0)^{-10} \bigg], \end{aligned} \tag{2.108}$$

we note that the leading few long wavelength terms are independent of $\bar{c}_1$. The gauge-invariant fractional energy density perturbation two-point function is

$$\begin{aligned} \langle \Delta_\Phi(k)\, \Delta_\Phi^*(k) \rangle = \frac{16\pi}{m_p^2}\, \epsilon^2 \bigg[& \frac{1}{18}\, {H_d}^{-2}\, T^6\, k\, (a_0)^{-4} + \frac{1}{2}\, \bar{c}_1\, T^3\, k^{-1}\, (a_0)^{-5} \\ & + \frac{9}{8}\, {\bar{c}_1}^2\, {H_d}^2\, k^{-3}\, (a_0)^{-6} - \frac{1}{6}\, \bar{c}_1\, {H_d}^{-2}\, T^3\, k\, (a_0)^{-7} \\ & + \frac{3}{8}\, {\bar{c}_1}^2\, k^{-1}\, (a_0)^{-8} + \frac{1}{8}\, {\bar{c}_1}^2\, {H_d}^{-2}\, k\, (a_0)^{-10} \bigg], \end{aligned} \tag{2.109}$$

again, the leading large–scale term does not depend on $\bar{c}_1$; it also agrees with eqn. (3.56) of Ref. [1] when $T = 1$, as it must. It is interesting to note that this term is scale-invariant, i.e., it is proportional to k, [8], as it must be (for geometrical reasons, [6,7]); we do not yet physically understand why the scale-invariant term is the third term in eqn. (2.108), or what the physical significance of the first two terms are.

3. Large Time

We shall now use the results of the previous section to determine the form of spatial irregularities in the radiation dominated epoch and the present, matter dominated, epoch. We assume that at the end of inflation the scalar field decays rapidly to form products that are quickly thermalized to behave like radiation (the time scales involved are assumed to be fast compared to the relevant cosmological time scales), so that the scalar field does not play a significant role in the large time evolution of the universe. We also assume that one can always focus on only one kind of matter — the scalar field in the scalar field dominated epoch, radiation during radiation dominance and nonrelativistic matter during baryon dominance (i.e., we ignore perturbations in all but the dominant form of matter). Following Ref. [1] we model the transition from scalar field dominance to radiation dominance and the transition from radiation dominance to baryon dominance as instantaneous, occurring at different values of synchronous gauge time in different spatial regions, whenever the local energy density drops to an appropriate value. More precisely, at a transition the equation of state (and the pressure) are changed discontinuously on the transition spatial hypersurface. This hypersurface differs from the corresponding synchronous gauge constant time hypersurface, however, spatial gradients in the local energy density are of first order in the perturbations so these hypersurfaces coincide to zeroth order in the perturbations and we join the scale factor and the homogeneous part of the energy density on the corresponding synchronous gauge constant time hypersurfaces.

The constants of integration in the expressions for the perturbations during the radiation and matter dominated epochs (given in Ref. [5]) are determined from the expressions for the perturbations in the scalar field dominated epoch by requiring that the linearized equations of covariant conservation of stress-energy and the linearized Einstein equations do not become singular on the transition spatial hypersurfaces. Following Sec. 2 of Ref. [4], the linearized equations of motion we use are those of a general fluid model (with an arbitrary, spacetime-dependent, "speed of sound", c_s). The first order (in spatial irregularity) equations of this model include, as special cases, both the scalar field model perturbation equations of motion as well as the ideal fluid (constant speed of sound) model perturbation equations of motion. This linearized model provides a general setting for deriving joining conditions for both the scalar field to radiation transition and the more conventional radiation–baryon transition.

We emphasize that our model of the transitions is only an approximation; in particular, the transitions are not really instantaneous and one must also account for effects arising from the scalar field–radiation coupling at the reheating transition and the radiation–baryon coupling at the second transition. We believe that the neglect of this small-scale physics does not significantly affect the evolution of the large-scale quantities which we are interested in. We plan to quantitatively examine this approximation in due course of time. Since the model incorrectly describes the relevant small-scale physics, we must focus on spatial momentum modes on scales larger than (or comparable to) the Hubble scale during the transitions. We shall use the complete expressions for spatial irregularities when determining the form of the constants of integration after a transition and then approximate these expressions for the constants of integration by focussing on their low spatial momentum part. This is in contrast to a number of earlier analyses where the expressions for spatial irregularities are first approximated by their low spatial momentum parts which are then used to determine the constants of integration after a transition. Although it might seem reasonable to suspect that these procedures commute, this has not yet been proved, so, at a risk of assigning undue physical significance to the instantaneous transition model (which unfortunately seems to be the only presently available model), we use the complete solutions of the differential equations when determining the constants of integration, rather than the approximate solutions.

Once the constants of integration in the expressions for the perturbations in the baryon dominated epoch have been determined, current observations may be used to constrain the free parameters of the model. To focus on the characteristics of the inflation epoch it is useful to specialize the model by requiring that the present universe be baryon dominated and spatially flat (so $\Omega_{\rm now} = 1$), that the present value of the Hubble parameter $H_{\rm now} = 100\,h\,{\rm km~s^{-1}Mpc^{-1}}$ (this determines the present value of the baryon energy density upto a factor of h^2) and that the parameter z_{BR} (the redshift of the radiation-baryon transition) has the conventional hot big bang value, $z_{BR} = 4\times 10^4\,h^2$ (this fixes the numerical value of the energy density in the radiation dominated epoch). The model is then specified by the values of the following parameters: $z_{R\Phi}$ (the redshift of the scalar field–radiation transition), $\bar\Phi$ [the minimum value of the scalar field for which the approximation (1.4) is valid], Φ_{b0i} and Φ_{b1i} [eqn. (2.9), the initial values of the order ϵ^0 and ϵ parts of the background scalar field], the order ϵ^0 part of the initial value of the time derivative of the background scalar field $\dot\Phi_{b0i}$ (which we have set to zero), $\dot\Phi_{b1i}$ (the initial value of the order ϵ part of the time derivative of the background scalar field) and $\bar c_2$ [eqn. (2.10), the initial value of the order ϵ part of the scale factor]. Fortunately, we find that the large time expressions for the perturbations do not depend on all of these parameters — a more relevant list of free parameters is given below.

3.1 Determining the Constants of Integration

The spatially homogeneous local energy density transition hypersurface and the corresponding constant time synchronous gauge hypersurface coincide to lowest order in the perturbations, so we match the scale factor and the spatially homogeneous part of the energy density on the appropriate constant time synchronous gauge hypersurface.

Matching the scale factor at $t_{R\Phi}$, the time of equal radiation and homogeneous scalar field energy density, we find in the radiation dominated epoch

$$a(t) = a_1[1 + M_R(t - t_{R\Phi})]^{1/2}, \tag{3.1}$$

where we have defined

$$a_1 = a_{0i}\, e^{H_d(t_{R\Phi}-t_i)} \left[1 + \epsilon \left\{ \bar{c}_2 - \frac{1}{8} H_d T^4 (t_{R\Phi} - t_i) \right\}\right],$$

and

$$M_R = 2 \left(\frac{8\pi}{3m_p^2}\, \rho_{bR}^{(1)} \right)^{1/2} .$$

The evolution of the scale factor in the scalar field dominated epoch is described by eqn. (2.10) and $\rho_{bR}^{(1)}$ is the value of the spatially homogeneous energy density (in radiation) at the scalar field–radiation transition when $a = a_1$. Similarly we find, in the baryon dominated epoch,

$$a(t) = a_2[1 + M_B(t - t_{BR})]^{2/3}, \tag{3.2}$$

where t_{BR} is the time of equal baryon and radiation energy density,

$$a_2 = a_{0i}\, e^{H_d(t_{R\Phi}-t_i)} \left[1 + \epsilon \left\{ \bar{c}_2 - \frac{1}{8} H_d T^4 (t_{R\Phi} - t_i) \right\}\right] [1 + M_R(t_{BR} - t_{R\Phi})]^{1/2}$$

and

$$M_B = \frac{3}{2} \left(\frac{8\pi}{3m_p^2}\, \rho_{bB}^{(2)} \right)^{1/2} ,$$

where $\rho_{bB}^{(2)}$ is the value of the energy density (in baryons) at the radiation–matter transition when $a = a_2$.

For future reference, we note that the redshift in the baryon dominated epoch is given by

$$1 + z = \frac{[1 + M_B(t_{\rm now} - t_{BR})]^{2/3}}{[1 + M_B(t - t_{BR})]^{2/3}}, \tag{3.3}$$

where $t_{\rm now}$ is the present value of t, while the redshift in the radiation dominated epoch is

$$1+z=\frac{[1+M_R(t_{BR}-t_{R\Phi})]^{1/2}}{[1+M_R(t-t_{R\Phi})]^{1/2}}[1+M_B(t_{\rm now}-t_{BR})]^{2/3}, \tag{3.4}$$

and that in the scalar field dominated epoch is given by

$$\begin{aligned}1+z=&e^{-H_d(t-t_{R\Phi})}\left[1+\frac{1}{8}\,\epsilon\,H_dT^4(t-t_{R\Phi})\right]\\ &\times[1+M_R(t_{BR}-t_{R\Phi})]^{1/2}[1+M_B(t_{\rm now}-t_{BR})]^{2/3},\end{aligned} \tag{3.5}$$

where we have worked to lowest order in ϵ. We note that this expression is independent of $\bar{c}_2$.

We also require that the homogeneous parts of the energy densities match at the transitions,

$$\rho_{b\Phi}(t_{R\Phi})=\rho_{bR}(t_{R\Phi}), \tag{3.6}$$

as well as

$$\rho_{bR}(t_{BR})=\rho_{bB}(t_{BR}). \tag{3.7}$$

Having chosen to use observational evidence to fix $H_{\rm now}$ and z_{BR} (upto a factor of h^2), eqn. (3.7) determines the value of the Hubble parameter during radiation dominance (given the evolution of the scale factor). Since $z_{R\Phi}$ has not yet been fixed, eqn. (3.6) only determines the inflation epoch Hubble parameter in terms of $z_{R\Phi}$.

To determine the constants of integration in the expressions for the perturbations during the radiation and matter dominated epochs we require that the equations of covariant conservation of stress-energy and Einstein's equations (in a general spacetime-dependent "speed of sound" fluid model) remain singularity-free on a spatially homogeneous local energy density transition hypersurface. The resulting set of joining conditions at the transition are derived in Sec. 3 of Ref. [4]. For a transition at a time t_{BR} (on the corresponding synchronous gauge spatial hypersurface) from a pretransition cosmological model dominated by a "fluid" denoted by R to a posttransition cosmological model dominated by a "fluid" denoted by B, where we leave unspecified the coefficients in the fluid equations of state and the "speed of sound" in these fluids, the joining conditions are

$$\delta_B(t_{BR},\vec{x}) = \frac{[\rho_{bB}(t_{BR}) + p_{bB}(t_{BR})]}{[\rho_{bR}(t_{BR}) + p_{bR}(t_{BR})]}\,\delta_R(t_{BR},\vec{x}), \tag{3.8}$$

$$\begin{aligned}[\rho_{bB}(t_{BR}) + p_{bB}(t_{BR})]\,v^i_B(t_{BR},\vec{x}) =& [\rho_{bR}(t_{BR}) + p_{bR}(t_{BR})]\,v^i_R(t_{BR},\vec{x}) \\ &+ \frac{1}{3}\frac{\rho_{bR}(t_{BR})}{a(t_{BR})H(t_{BR})}\,\frac{[p_{bB}(t_{BR}) - p_{bR}(t_{BR})]}{[\rho_{bR}(t_{BR}) + p_{bR}(t_{BR})]}\,\partial_i\delta_R(t_{BR},\vec{x}),\end{aligned} \tag{3.9}$$

$$h^{(B)}_{22}(t_{BR},\vec{k}) + h^{(B)}_{33}(t_{BR},\vec{k}) = h^{(R)}_{22}(t_{BR},\vec{k}) + h^{(R)}_{33}(t_{BR},\vec{k}), \tag{3.10}$$

$$h^{(B)}_{I1}(t_{BR},\vec{k}) = h^{(R)}_{I1}(t_{BR},\vec{k}), \tag{3.11}$$

$$\begin{aligned}\dot{h}^{(B)}_{22}(t_{BR},\vec{x}) + \dot{h}^{(B)}_{33}(t_{BR},\vec{x}) =& \dot{h}^{(R)}_{22}(t_{BR},\vec{x}) + \dot{h}^{(R)}_{33}(t_{BR},\vec{x}) \\ &+ 2H(t_{BR})\,\frac{[p_{bR}(t_{BR}) - p_{bB}(t_{BR})]}{[\rho_{bR}(t_{BR}) + p_{bR}(t_{BR})]}\delta_R(t_{BR},\vec{x}),\end{aligned} \tag{3.12}$$

$$\dot{h}^{(B)}_{I1}(t_{BR},\vec{x}) = \dot{h}^{(R)}_{I1}(t_{BR},\vec{x}), \tag{3.13}$$

$$\begin{aligned}h^{(B)}(t_{BR},\vec{k}) = h^{(R)}(t_{BR},\vec{k}) &- 4\left(\frac{k}{a(t_{BR})H(t_{BR})}\right)^2 \frac{[p_{bR}(t_{BR}) - p_{bB}(t_{BR})]}{[\rho_{bR}(t_{BR}) + p_{bR}(t_{BR})]} \\ &\times \frac{\rho_{bR}(t_{BR})}{[\rho_{bR}(t_{BR}) - 3p_{bR}(t_{BR})]}\,\frac{\rho_{bR}(t_{BR})}{[\rho_{bB}(t_{BR}) - 3p_{bB}(t_{BR})]}\,\delta_R(t_{BR},\vec{k}),\end{aligned} \tag{3.14}$$

$$\dot{h}^{(B)}(t_{BR},\vec{k}) = \dot{h}^{(R)}(t_{BR},\vec{k}) + 3H(t_{BR})\,\frac{[p_{bR}(t_{BR}) - p_{bB}(t_{BR})]}{[\rho_{bR}(t_{BR}) + p_{bR}(t_{BR})]}\,\delta_R(t_{BR},\vec{k}), \tag{3.15}$$

$$\begin{aligned}\dot{\delta}_B(t_{BR},\vec{k}) =& \dot{\delta}_R(t_{BR},\vec{k}) \\ &- \frac{[p_{bR}(t_{BR}) - p_{bB}(t_{BR})]}{[\rho_{bR}(t_{BR}) + p_{bR}(t_{BR})]}\left[\frac{[\rho_{bR}(t_{BR}) + p_{bR}(t_{BR})]}{2\rho_{bR}(t_{BR})}\,\dot{h}^{(R)}(t_{BR},\vec{k})\right. \\ &\quad + H(t_{BR})\,\delta_R(t_{BR},\vec{k})\left\{\frac{1}{3}\left(\frac{k}{a(t_{BR})H(t_{BR})}\right)^2\right. \\ &\quad\quad \left.\left. + \frac{3}{2}\,\rho^{-1}_{bR}(t_{BR})\,[\rho_{bR}(t_{BR}) + 2p_{bR}(t_{BR}) + p_{bB}(t_{BR})]\right\}\right]\end{aligned} \tag{3.16}$$

$$- \frac{3H(t_{BR})\delta_R(t_{BR})}{[\rho_{bR}(t_{BR})+p_{bR}(t_{BR})]} \left\{ c_{sB}^2[\rho_{bB}(t_{BR})+p_{bB}(t_{BR})] - c_{sR}^2[\rho_{bR}(t_{BR})+p_{bR}(t_{BR})] \right\} ,$$

$$h_i^{(B)}(t_{BR},\vec{x}) = h_i^{(R)}(t_{BR},\vec{x}), \tag{3.17}$$

$$\dot{h}_i^{(B)}(t_{BR},\vec{x}) = \dot{h}_i^{(R)}(t_{BR},\vec{x}). \tag{3.18}$$

These equations are synchronous gauge equations — all quantities (fields, coordinates, spatial momenta and derivatives) are synchronous gauge quantities. When describing what these equations do we shall assume some familiarity with the results and analysis of Ref. [5]. δ_B, v_B^1 and $h_{22}^{(B)}+h_{33}^{(B)}$ depend on three constants of integration, $c_2^{(B)}$ and $c_\pm^{(B)}$; these constants of integration are determined by eqn. (3.8), by the $i=1$ part of eqns. (3.9) and by eqn. (3.10). The $i=I$ components of eqns. (3.9) along with eqns. (3.11) are four joining conditions for the four constants of integration, $c_{I,1}^{(B)}$ and $c_{I,2}^{(B)}$, in the solutions for v_B^I and $h_{I1}^{(B)}$. Eqn. (3.12) is a consistency equation — it has to be identically satisfied by the expressions for $c_\pm^{(B)}$ and $c_2^{(B)}$ derived from eqns. (3.8) – (3.10) and will provide a check on our algebraic manipulations. In a similar manner, eqn. (3.13) is also a consistency condition — the expressions for $c_{I,2}^{(B)}$ must identically satisfy it. Eqn. (3.14) determines $c_1^{(B)}$ in the expression for the trace of the metric perturbation, $h^{(B)}$, it must be modified if either of the fluids is radiation (see the discussion below eqn. (3.59) in Ref. [4]); we shall ignore this equation in what follows since we do not have need for $c_1^{(B)}$. Eqns. (3.15) and (3.16) are further consistency equations. The four constants of integration which correspond to gravitational wave perturbations, $c_{i1}^{(B)}$ and $c_{i2}^{(B)}$ (where $i=+,\times$), are determined by eqns. (3.17) and (3.18). The consistency conditions, which are a consequence of requiring that the perturbation equations of motion not be singular on the transition hypersurface, also follow from the perturbation equations of motion and the other joining conditions — they contain no new information but serve to provide a check on our algebraic manipulations. Finally, this set of joining conditions is complete (see Sec. 4 of Ref. [4]).

One might wish to repackage a subset of these joining conditions in a gauge-invariant form. We have done this in Sec. 4 of Ref. [4]. Along with the gauge-invariant joining conditions (3.17) and (3.18), which determine the four gauge-invariant constants of integration $c_{i1}^{(B)}$ and $c_{i2}^{(B)}$, we have

$$\Delta_B(t_{BR},\vec{k}) = \Delta_R(t_{BR},\vec{k}), \tag{3.19}$$

$$\frac{A_B(t_{BR},\vec{k})}{[\rho_{bB}(t_{BR})+p_{bB}(t_{BR})]} = \frac{A_R(t_{BR},\vec{k})}{[\rho_{bR}(t_{BR})+p_{bR}(t_{BR})]}\,, \tag{3.20}$$

where the gauge-invariant variables, Δ_Φ and A_Φ, are defined in eqns. (2.75) and (2.76). Eqns. (3.19) and (3.20) provide two joining conditions for the gauge-invariant constants of integration $c_\pm^{(B)}$.

Following Sec. 4.1 of Ref. [1], we match the exact expressions for irregularities (Sec. 2 above and Appendices C and B of Ref. [5]) at $t_{R\Phi}$ and t_{BR} to determine the constants of integration and then derive approximate expressions valid for wavelengths that were large compared to the Hubble radius during the transitions.

We first consider the scalar field dominance to radiation dominance transition. Using the joining conditions for δ, $(\rho_b+p_b)v^1$ and $h_{22}+h_{33}$ [eqns. (3.8), (3.9) and (3.10)] we find, in the radiation dominated epoch, that the fractional energy density perturbation is

$$\delta_R(x) = c_2^{(R)}e^{-x} - 2{\lambda_R}^{-5/2}e^{-x/2}(c_+^{(R)}G_1^R(x) + c_-^{(R)}G_2^R(x)), \tag{3.21}$$

where we have defined

$$G_1^R(x) = \left(\frac{2}{\pi}\right)^{1/2}\left[i\lambda_R\, e^{x/2} - 2 - 2i{\lambda_R}^{-1}e^{-x/2}\right]\exp\left(i\lambda_R\, e^{x/2}\right) \tag{3.22}$$

[and $G_2^R = (G_1^R)^*$], that the longitudinal peculiar velocity perturbation is

$$v_R^1(x) = -i\lambda_R\frac{\sqrt{3}}{2}\left[-\frac{1}{2}\,c_2^{(R)}e^{-x/2} + {\lambda_R}^{-5/2}(c_+^{(R)}\widetilde{G}_1^R(x) + c_-^{(R)}\widetilde{G}_2^R(x))\right], \tag{3.23}$$

where

$$\widetilde{G}_1^R(x) = -\left(\frac{2}{\pi}\right)^{1/2}\left[1 + 2i{\lambda_R}^{-1}e^{-x/2}\right]\exp\left(i\lambda_R\, e^{x/2}\right) \tag{3.24}$$

[and $\widetilde{G}_2^R = (\widetilde{G}_1^R)^*$], and

$$h_{22}^{(R)}(x) + h_{33}^{(R)}(x) = c_2^{(R)} e^{-x} + 2\, e^{3x/4}(c_+^{(R)} \widetilde{F}_1^R(x) + c_-^{(R)} \widetilde{F}_2^R(x)), \tag{3.25}$$

where we have defined

$$\widetilde{F}_1^R(x) = 2i \left(\frac{2}{\pi}\right)^{1/2} {\lambda_R}^{-7/2} e^{-7x/4} \exp\left(i\lambda_R\, e^{x/2}\right) \tag{3.26}$$

[and $\widetilde{F}_2^R = (\widetilde{F}_1^R)^*$]. We note that the gauge-invariant measure of energy density perturbations is given by

$$\begin{aligned}\Delta_R(x) = \left(\frac{8}{\pi}\right)^{1/2} {\lambda_R}^{-5/2} e^{-x/2} \Big[& c_+^{(R)} \left(1 - i\lambda_R\, e^{x/2}\right) \exp\left(i\lambda_R\, e^{x/2}\right) \\ & + c_-^{(R)} \left(1 + i\lambda_R\, e^{x/2}\right) \exp\left(-i\lambda_R\, e^{x/2}\right) \Big].\end{aligned} \tag{3.27}$$

The last factor in eqn. (3.22) [as well as the similar factors in eqn. (3.27)] means that small-scale energy density irregularities oscillate during the radiation dominated epoch.

The integration constants in these formulae are given by (where we have worked to lowest order in ϵ)

$$c_2^{(R)} = \frac{1}{\epsilon} \left(\frac{16\pi}{m_p^2}\right)^{1/2} 32\sqrt{2}\, {\lambda_R}^{-2}\, {H_d}^2\, k^{-3/2}\, \frac{G_1(w_{R\Phi})}{[A_1 {w_{R\Phi}}^3 + A_2]^2}\,, \tag{3.28}$$

and

$$\begin{aligned}c_\pm^{(R)} = & \frac{1}{\epsilon} \left(\frac{16\pi}{m_p^2}\right)^{1/2} 4\sqrt{\pi}\, {\lambda_R}^{3/2}\, {H_d}^2\, k^{-3/2}\, \frac{G_1(w_{R\Phi})}{[A_1 {w_{R\Phi}}^3 + A_2]^2} \\ & \times \left[\pm i - \lambda_R\, e^{x_R/2}\right] \exp\left(\mp i\lambda_R\, e^{x_R/2}\right) .\end{aligned} \tag{3.29}$$

In these expressions, G_1 has been defined in eqn. (2.52), A_1 and A_2 are given in eqns. (2.22) and (2.23) and we have defined

$$\begin{aligned} w_{R\Phi} &= \frac{k}{a(t_{R\Phi}) H_d}\,, \\ e^x &= t - t_{R\Phi} + {M_R}^{-1}, \\ e^{x_R} &= {M_R}^{-1}, \\ \lambda_R &= \frac{2}{\sqrt{3}} \frac{k}{a_1} {M_R}^{-1/2}. \end{aligned}$$

It may be verified that the consistency equations (3.12), (3.15) and (3.16), [19], are satisfied at the scalar field–radiation transition. The expressions for $c_\pm^{(R)}$, eqns. (3.29), may also be derived by matching the gauge-invariant quantities Δ and $A/(\rho_b + p_b)$ at the transition, eqns. (3.19) and (3.20).

The radiation–baryon transition has been studied in Sec. 4.1 of Ref. [1]; we record the relevant formulae here. Using the joining conditions for δ, $(\rho_b+p_b)v^1$ and $h_{22}+h_{33}$, we have, in the baryon dominated epoch,

$$\delta_B(x) = c_2^{(B)} e^{-x} + c^{(B)} e^{2x/3}, \tag{3.30}$$

as well as

$$v_B^1(x) = i\,\frac{3}{8}\frac{k}{a_2} {M_B}^{-2/3} c_8^{(B)} e^{-2x/3}, \tag{3.31}$$

and

$$h_{22}^{(B)}(x) + h_{33}^{(B)}(x) = -\frac{40}{9}\frac{a_2^2}{k^2} {M_B}^{4/3} c^{(B)} + c_8^{(B)} e^{-x}, \tag{3.32}$$

where we have defined

$$e^x = t - t_{BR} + {M_B}^{-1}.$$

The constant of integration $c_2^{(B)}$ in these expressions is related to the constants of integration in the radiation dominated epoch solutions through

$$c_2^{(B)} = c_2^{(R)} - 2{\lambda_R}^{-5/2} e^{\bar{x}_B/2} \left(c_+^{(R)} \bar{G}_1^R(\bar{x}_B) + c_-^{(R)} \bar{G}_2^R(\bar{x}_B) \right), \tag{3.33}$$

where we have defined

$$\begin{aligned} \bar{G}_1^R(\bar{x}_B) = \left(\frac{2}{\pi}\right)^{1/2} \Big[&- 2i\,{\lambda_R}^{-1} e^{-\bar{x}_B/2} - 2 + i\,\lambda_R\, e^{\bar{x}_B/2} + \frac{4}{15}{\lambda_R}^2 e^{\bar{x}_B} \\ &+ \frac{2}{15} i\,{\lambda_R}^3 e^{3\bar{x}_B/2} \Big] \exp\left(i\lambda_R\, e^{\bar{x}_B/2} \right) \end{aligned} \tag{3.34}$$

[and $\bar{G}_2^R = (\bar{G}_1^R)^*$]. The constant of integration $c^{(B)}$, corresponding to a gauge-invariant solution, is related to $c_\pm^{(R)}$ by

$$c^{(B)} = \frac{3}{5}\left(\frac{2}{\pi}\right)^{1/2} \widetilde{\lambda}_B{}^{-1/2} e^{-x_B/12}\left[c_+^{(R)}\left(1+\frac{i}{2}\lambda_R\, e^{\bar{x}_B/2}\right)\exp\left(i\lambda_R\, e^{\bar{x}_B/2}\right)\right.$$
$$\left. + c_-^{(R)}\left(1-\frac{i}{2}\lambda_R\, e^{\bar{x}_B/2}\right)\exp\left(-i\lambda_R\, e^{\bar{x}_B/2}\right)\right], \quad (3.35)$$

while the constant of integration $c_8^{(B)}$ is given by

$$c_8^{(B)} = \frac{4}{3}\, c_2^{(R)} - \frac{16}{9}\lambda_R{}^{-5/2} e^{\bar{x}_B/2}\left(c_+^{(R)}\hat{G}_1^R(\bar{x}_B) + c_-^{(R)}\hat{G}_2^R(\bar{x}_B)\right), \quad (3.36)$$

where

$$\hat{G}_1^R(\bar{x}_B) = \left(\frac{2}{\pi}\right)^{1/2}\left[-\frac{i}{2}\lambda_R\, e^{\bar{x}_B/2} - 1 - 3i\,\lambda_R{}^{-1} e^{-\bar{x}_B/2}\right]\exp\left(i\lambda_R\, e^{\bar{x}_B/2}\right) \quad (3.37)$$

[and $\hat{G}_2^R = (\hat{G}_2^R)^*$]. In these formulae the constants $c_2^{(R)}$ and $c_\pm^{(R)}$ are given in eqns. (3.28) and (3.29) and we have defined

$$\begin{aligned} e^{x_B} &= M_B{}^{-1}, \\ e^{\bar{x}_B} &= \frac{3}{4}\, e^{x_B}, \\ \widetilde{\lambda}_B &= 3k\,(a_2 M_B{}^{2/3})^{-1}. \end{aligned}$$

It has been verified, in Sec. 4.1 of Ref. [1], that the consistency conditions (3.12), (3.15) and (3.16) are satisfied at the radiation–baryon transition. We note that unlike the exponential potential scalar field case, the constants of integration corresponding to the time translation invariance solution in the radiation epoch, $c_2^{(R)}$ [eqn. (3.28)], and in the baryon epoch, $c_2^{(B)}$ [eqn. (3.33)], are fixed — this is because the inflation epoch time translation invariance solution is absent at lowest order in ϵ in eqns. (2.51) and (2.56) — see the discussion in the paragraph following eqn. (4.35) and in the paragraph preceding eqn. (4.44) in Ref. [1].

In the baryon dominated epoch the gauge-invariant measure of energy density perturbations, Δ_B, is given by

$$\Delta_B(x) = \left(c_2^{(B)} - \frac{3}{4}c_8^{(B)}\right)e^{-x} + c^{(B)} e^{2x/3}, \quad (3.38)$$

where

$$c_2^{(B)} - \frac{3}{4}c_8^{(B)} = \frac{8}{3}\,\lambda_R{}^{-5/2}e^{\bar{x}_B/2}\left(c_+^{(R)}\check{G}_1^R(\bar{x}_B) + c_-^{(R)}\check{G}_2^R(\bar{x}_B)\right), \tag{3.39}$$

and we have defined

$$\check{G}_1^R(\bar{x}_B) = \left(\frac{2}{\pi}\right)^{1/2}\left[1 - i\,\lambda_R\,e^{\bar{x}_B/2} - \frac{1}{5}\lambda_R{}^2e^{\bar{x}_B} - \frac{i}{10}\,\lambda_R{}^3e^{3\bar{x}_B/2}\right]\exp\left(i\lambda_R\,e^{\bar{x}_B/2}\right) \tag{3.40}$$

and $\check{G}_2^R = (\check{G}_1^R)^*$. Eqns. (3.35) and (3.39) may also be derived by matching Δ and $A/(\rho_b+p_b)$ at the transition. Unlike eqns. (3.22) and (3.27), fractional energy density irregularities in the matter dominated epoch, eqns. (3.30) and (3.38), do not oscillate on small scales.

One may use the joining conditions for the trace of the metric perturbation to determine the constant of integration $c_1^{(R)}$ in the radiation dominated epoch and the constant of integration $c_1^{(B)}$ in the baryon dominated epoch; we do not do this since we shall not need the explicit form of the trace of the metric perturbation h.

The joining conditions for the transverse peculiar velocity perturbations and the h_{I1} part of the metric perturbations are given in eqns. (3.9) and (3.11). Using eqns. (2.55) and (3.9) [as well as eqns. (C9) and (B16) of Ref. [5]] we find $v_R^I = 0 = v_B^I$ or $c_{I,2}^{(R)} = 0 = c_{I,2}^{(B)}$. From eqns. (2.31) and (3.11) [as well as eqns. (C8) and (B15) of Ref. [5]] we find $h_{I1}^{(R)} = h_{I1} = h_{I1}^{(B)}$ or $c_{I,1}^{(R)} = c_{I,1} = c_{I,1}^{(B)}$. These expressions for the constants of integration satisfy the consistency conditions (3.13). We note that scalar field dominated models of inflation do not generate transverse peculiar velocity perturbations, [1] (at least in the linear perturbation approximation).

Using the joining conditions for gravitational wave perturbations, eqns. (3.17) and (3.18), the inflation epoch expressions for gravitational wave perturbations, eqns. (2.32) and (2.84), as well as the radiation epoch form of these perturbations, eqns. (39) and (40) of Ref. [5], we find, in the radiation dominated epoch, that the gravitational wave perturbations are

$$h_i^{(R)} = e^{-x/4}\left\{c_{i1}^{(R)}H_{-1/2}^{(1)}\left(\widetilde{\lambda}_R\,e^{x/2}\right) + c_{i2}^{(R)}H_{-1/2}^{(2)}\left(\widetilde{\lambda}_R\,e^{x/2}\right)\right\}, \tag{3.41}$$

where $\widetilde{\lambda}_R = \sqrt{3}\lambda_R$ and we have defined

$$c_{i1}^{(R)} = \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{\sqrt{\pi}}{2}\, \widetilde{\lambda}_R{}^{-1/2} H_d\, k^{-3/2}\,, \tag{3.42}$$

$$\begin{aligned} c_{i2}^{(R)} &= \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{\sqrt{\pi}}{2}\, \widetilde{\lambda}_R{}^{-1/2} H_d\, k^{-3/2} \exp\left(2i\,\widetilde{\lambda}_R\, e^{x_R/2}\right) \\ &\quad \times \left(-1 + 2i\,\widetilde{\lambda}_R\, e^{x_R/2} + 2\,\widetilde{\lambda}_R{}^2\, e^{x_R}\right)\,. \end{aligned} \tag{3.43}$$

In the matter dominated epoch we have, from eqns. (4.49) and (4.50) of Ref. [1],

$$h_i^{(B)} = e^{-x/2}\left[c_{i1}^{(B)} H_{-3/2}^{(1)}\left(\widetilde{\lambda}_B\, e^{x/3}\right) + c_{i2}^{(B)} H_{-3/2}^{(2)}\left(\widetilde{\lambda}_B\, e^{x/3}\right)\right], \tag{3.44}$$

where the constants of integration are related to the radiation epoch constants of integration through

$$\begin{aligned} c_{i1}^{(B)} &= i\,\frac{\pi}{4}\left(\frac{2}{\sqrt{3}}\right)^{1/2} \widetilde{\lambda}_B\ e^{7x_B/12} \\ &\quad \left[c_{i1}^{(R)}\left\{H_{-5/2}^{(2)}\left(\widetilde{\lambda}_B e^{x_B/3}\right) H_{-1/2}^{(1)}\left(\widetilde{\lambda}_R e^{\bar{x}_B/2}\right) - H_{-3/2}^{(2)}\left(\widetilde{\lambda}_B e^{x_B/3}\right) H_{-3/2}^{(1)}\left(\widetilde{\lambda}_R e^{\bar{x}_B/2}\right)\right\}\right. \\ &\quad \left. + c_{i2}^{(R)}\left\{H_{-5/2}^{(2)}\left(\widetilde{\lambda}_B e^{x_B/3}\right) H_{-1/2}^{(2)}\left(\widetilde{\lambda}_R e^{\bar{x}_B/2}\right) - H_{-3/2}^{(2)}\left(\widetilde{\lambda}_B e^{x_B/3}\right) H_{-3/2}^{(2)}\left(\widetilde{\lambda}_R e^{\bar{x}_B/2}\right)\right\}\right], \end{aligned} \tag{3.45}$$

$$\begin{aligned} c_{i2}^{(B)} &= -i\,\frac{\pi}{4}\left(\frac{2}{\sqrt{3}}\right)^{1/2} \widetilde{\lambda}_B\ e^{7x_B/12} \\ &\quad \left[c_{i1}^{(R)}\left\{H_{-5/2}^{(1)}\left(\widetilde{\lambda}_B e^{x_B/3}\right) H_{-1/2}^{(1)}\left(\widetilde{\lambda}_R e^{\bar{x}_B/2}\right) - H_{-3/2}^{(1)}\left(\widetilde{\lambda}_B e^{x_B/3}\right) H_{-3/2}^{(1)}\left(\widetilde{\lambda}_R e^{\bar{x}_B/2}\right)\right\}\right. \\ &\quad \left. + c_{i2}^{(R)}\left\{H_{-5/2}^{(1)}\left(\widetilde{\lambda}_B e^{x_B/3}\right) H_{-1/2}^{(2)}\left(\widetilde{\lambda}_R e^{\bar{x}_B/2}\right) - H_{-3/2}^{(1)}\left(\widetilde{\lambda}_B e^{x_B/3}\right) H_{-3/2}^{(2)}\left(\widetilde{\lambda}_R e^{\bar{x}_B/2}\right)\right\}\right]. \end{aligned} \tag{3.46}$$

Small–scale gravitational wave perturbations oscillate in the radiation dominated epoch, eqn. (3.41), and in the baryon dominated epoch, eqn. (3.44).

3.2 Large-Scale Approximation and Expressions for Irregularities

The large time characteristics of the inflation epoch model are most easily examined if we focus on perturbations that were on scales larger than the Hubble scale during the transition from scalar field dominance to radiation dominance and the transition from radiation dominance to baryon dominance. These large-scale perturbations are the baryon dominated epoch expressions of the previous section where only the leading asymptotic terms of the expressions for the constants of integration are retained in the limit when $k/[a(t_{R\Phi})H(t_{R\Phi})]$ and $k/[a(t_{BR})H(t_{BR})]$ are small.

For perturbations that were outside the Hubble radius during the scalar field–radiation transition, the dominant large-scale terms in the constants which characterize energy density, longitudinal peculiar velocity and spatial curvature perturbations in the radiation dominated epoch, eqns. (3.28) and (3.29), are

$$c_2^{(R)} = \frac{i}{\epsilon} \left(\frac{16\pi}{m_p^2}\right)^{1/2} 3\sqrt{2}\, T^{-3}\, k^{-3/2} \left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\right]^{-2} + \cdots, \tag{3.47}$$

and

$$c_{\pm}^{(R)} = \mp\frac{1}{\epsilon} \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{\sqrt{\pi}}{4} \frac{H_d}{T^3} \left(\frac{\lambda_R}{k}\right)^{3/2} + \cdots. \tag{3.48}$$

In the limit $T = 1$ these expressions agree with the $\epsilon \to 0$ limits of eqns. (5.1) and (5.2) of Ref. [1] (as they must). Retaining subleading terms in eqns. (3.47) and (3.48) and using

$$\frac{k}{a(t_{BR})H(t_{BR})} \gg \frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\,.$$

we find that the matter dominated epoch constants of integration given in eqns. (3.33), (3.35), (3.36) and (3.39) reduce to, for perturbations that enter the Hubble radius in the matter dominated epoch,

$$\begin{aligned} c_2^{(B)} = -\frac{i}{\epsilon} \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{1}{\sqrt{2}}\, T^{-3}\, k^{-3/2} & \\ \times \left\{1 - \frac{1}{20}\left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\right]^{-2} \left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^{4} + \cdots\right\}&, \end{aligned} \tag{3.49}$$

$$c^{(B)} = -\frac{i}{\epsilon} \left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{3}{20} \left(3\sqrt{2}\right)^{1/2} \frac{H_d}{T^3} \left(\frac{\lambda_R}{k}\right)^{3/2} \left(\frac{\widetilde{\lambda}_B}{k}\right)^{-1/4} k^{-1/4} \left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^{3/4} + \cdots, \tag{3.50}$$

and

$$c_8^{(B)} = -\frac{i}{\epsilon}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{2}{3}\,\frac{\sqrt{2}}{T^3}\,k^{-3/2} \times\left\{1-\frac{7}{108}\left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\right]^{-2}\left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^{4}+\cdots\right\}, \tag{3.51}$$

as well as

$$c_2^{(B)} - \frac{3}{4}c_8^{(B)} = -\frac{i}{\epsilon}\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{\sqrt{2}}{135}\,T^{-3}\,k^{-3/2}\left[\frac{k}{a(t_{R\Phi})H(t_{R\Phi})}\right]^{-2}\left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^{4}+\cdots. \tag{3.52}$$

In the $T = 1$ limit, eqns. (3.49) – (3.52) agree with the $\epsilon \to 0$ limit of eqns. (5.3) – (5.6) of Ref. [1].

For gravitational wave perturbations that were outside the Hubble radius during the scalar field–radiation transition, the constants of integration which characterize them in the radiation dominated era, eqns. (3.42) and (3.43), become

$$c_{i(\frac{1}{2})}^{(R)} = \pm\left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{\sqrt{\pi}}{2}\left(\frac{\widetilde{\lambda}_R}{k}\right)^{-1/2} H_d\,k^{-2}+\cdots. \tag{3.53}$$

We note that these expressions are independent of ϵ and T (as they must be) and that they coincide with eqns. (5.7) of Ref. [1] (as they must). For perturbations that enter the Hubble radius in the matter dominated epoch we find that eqns. (3.45) and (3.46) reduce to

$$c_{i(\frac{1}{2})}^{(B)} = \mp\left(\frac{16\pi}{m_p^2}\right)^{1/2}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{3}{2}\right)^{3/4} H_d \times\left(\frac{\widetilde{\lambda}_R}{k}\right)^{-1/2}\left(\frac{\widetilde{\lambda}_B}{k}\right)^{-3/4} k^{-11/4}\left[\frac{k}{a(t_{BR})H(t_{BR})}\right]^{-1/4}+\cdots; \tag{3.54}$$

again, these expressions are ϵ and T independent and coincide with eqn. (5.8) of Ref. [1] (in the $\epsilon \to 0$ limit).

We shall now use these asymptotic expansions of the constants of integration to determine the large time form of the large-scale peculiar velocity perturbation and the large time form of the power spectrum of large-scale energy density irregularities.

In the baryon dominated epoch the gauge-dependent longitudinal peculiar velocity perturbation decays linearly with the scale factor, eqn. (3.31),

$$v_B^1(k,t) = i\,\frac{3}{8}\,k\,c_8^{(B)}\,a^{-1}, \tag{3.55}$$

where $c_8^{(B)}$ is given by eqn. (3.51). Transverse peculiar velocity perturbations, which are gauge-invariant, vanish [see the discussion in the paragraph above eqn. (3.41)].

The power spectrum of gauge-invariant energy density irregularities, in the baryon dominated epoch, is given by

$$P_\Delta(k,t) \equiv \langle \Delta_B(k,t)\Delta_B^*(k,t)\rangle = \frac{1}{\epsilon^2 T^6}\,\frac{16\pi}{m_p^2}\,k\,\left[|A|^2\,a^2 + |B_\Delta|\,a^{-1/2} + |C_\Delta|^2\,a^{-3}\right], \tag{3.56}$$

where $|A|, |B_\Delta|$ and $|C_\Delta|$ correspond to gauge-invariant solutions and are given by

$$|A| = \frac{1}{20}\left(\frac{3}{\sqrt{2}}\right)^{3/2}\left(\frac{\lambda_R}{k}\right)^{7/2}\left(\frac{\widetilde{\lambda}_B}{k}\right)^{3/4}\left[a(t_{R\Phi})H(t_{R\Phi})\right]^2\left[a(t_{BR})H(t_{BR})\right]^{-3/4}, \tag{3.57}$$

$$|B_\Delta| = \frac{1}{50(2)^{1/4}}\left(\frac{\lambda_R}{k}\right)^{7/2}\left(\frac{\widetilde{\lambda}_B}{k}\right)^{-3/4}\left[a(t_{R\Phi})H(t_{R\Phi})\right]^4\left[a(t_{BR})H(t_{BR})\right]^{-19/4}, \tag{3.58}$$

$$|C_\Delta| = \frac{1}{15}\left(\frac{2}{3}\right)^{1/2}\left(\frac{\widetilde{\lambda}_B}{k}\right)^{-3/2}\left[a(t_{R\Phi})H(t_{R\Phi})\right]^2\left[a(t_{BR})H(t_{BR})\right]^{-4}. \tag{3.59}$$

The term proportional to $|A|^2$ in eqn. (3.56) is the growing adiabatic mode (and has the expected dependence on the scale factor). This term is also the dominant term in $\left\langle \delta_B(k)\,\delta_B^*(k)\right\rangle$. We note that in the limit $T=1$ eqns. (3.56) – (3.59) agree with eqns. (5.14) – (5.17) of Ref. [1]. The spatial momentum dependence of the power spectrum is of the scale-invariant form, [8], and does not have the extra factor, which is a function of $\ln(k)$, present in the results of earlier analyses. As expected, the power spectrum "diverges" like ϵ^{-2}. [We emphasize that given the approximations we have used, eqn. (3.56) describes the complete k and ϵ dependence of the power spectrum (i.e., the variables T and $z_{R\Phi}$ are, at this level, k and ϵ independent).]

The physical reason for the "divergence" of the power spectrum has been discussed in Ref. [1]; we recapitulate this discussion here. Even though δ_Φ (during inflation) obeys the standard assumptions of linear perturbation theory [it is proportional to ϵ, eqn. (2.51)], the spatially homogeneous energy density hypersurfaces in the almost de Sitter spacetime, eqn. (2.10), are exceedingly displaced from the corresponding synchronous gauge constant time hypersurfaces. This is because the extremely slow evolution of the background energy density (which corresponds to an almost constant cosmological constant) requires that one must transform to a hypersurface of substantially different time before the change in $\rho_{b\Phi}$ is large enough to compensate for δ_Φ. More precisely, during inflation the temporal displacement of the spatially homogeneous energy density hypersurface from the corresponding constant time (t_Φ) synchronous gauge hypersurface is

$$\Delta t\,(t_\Phi, \vec{k}) = -\,\frac{\rho_{b\Phi}(t_\Phi)}{\dot{\rho}_{b\Phi}(t_\Phi)}\,\delta_\Phi(t_\Phi, \vec{k}) \tag{3.60}$$

(see the discussion in the paragraph containing eqn. (5.19) in Ref. [1]). Now from eqns. (2.9), (2.49) and (2.50) we have

$$\dot{\rho}_{b\Phi} = -3H(\rho_{b\Phi} + p_{b\Phi}) = -3H_d\,\frac{m_p^2}{16\pi}(\dot{\Phi}_b)^2 \propto \epsilon^2 \tag{3.61}$$

and $\rho_{b\Phi}$ is independent of ϵ to lowest order; eqns. (2.51) and (3.60) then imply

$$\Delta t(t_\Phi, \vec{k}) \propto \frac{1}{\epsilon}\,. \tag{3.62}$$

This means that the spatial hypersurface on which the energy density is spatially homogeneous is not "close" to the corresponding synchronous gauge constant time hypersurface. Now the reheating transition is governed by local physics and must occur on a spatially homogeneous energy density hypersurface, so to join the perturbations we need to get the (prereheating) scalar field dominated epoch spatially homogeneous energy density hypersurface and the (postreheating) radiation dominated epoch spatially homogeneous energy density hypersurface to line up — this requires that, after the transition, δ_R be large, i.e., proportional to ϵ^{-1} (since during radiation dominance the constant time synchronous gauge hypersurfaces and the spatially homogeneous energy density hypersurfaces are not anomalously relatively displaced). If ϵ is exceedingly small the assumptions of linear perturbation theory are no longer valid during radiation dominance, see the discussion in Sec. 5.2 of Ref. [1]. We emphasize that the choice $c_0 = 0$ made in Sec. 2.1 [see eqn. (2.6)] does not invalidate our conclusion that $\Delta t \propto \epsilon^{-1}$ during inflation; this is because one requires that inflation last for at least 60 – 70 Hubble times while for reasonable values of c_0 and ϵ the qualitative behaviour of the solution for the background scalar field approaches eqn. (2.9) within half a dozen Hubble times.

The parameters which characterize the large time features of the inflation model ($z_{R\Phi}$, ϵ and T) are constrained by comparing the expressions for spatial inhomogeneities to observational data. (The large time expressions for the perturbations do not depend on the other parameters listed in the last paragraph of the introduction to Sec. 3, as a result they cannot be constrained in this manner.) To compare to observational data one must transform to the appropriate coordinate system (see the discussion in Sec. 5.4 of Ref. [1]); we shall not do this here since it is clear that, at present, the model has too many (three) free parameters to be usefully constrained by observational data. We have shown, in Ref. [1], that even the two parameter exponential potential inflation model is not obviously inconsistent with observational data for a fairly wide range of values of the parameters. It might be thought that without comparing to observational data one cannot justify the assumption that ϵ is small — this is partially incorrect because in the $T = 1$ limit the expressions found here reduce to the $\epsilon \to 0$ limit expressions of the model studied in Ref. [1] where it was shown that values of $\epsilon \sim 10^{-8}$ (or even smaller) are not obviously inconsistent with the observations. We hope to eventually return to a comparison of this model with observational data once T has been theoretically estimated — some comments on how this might be accomplished are given in the Conclusion.

The expressions for the gravitational wave perturbation two-point function and the energy density carried by these spatial irregularities are given in Sec. 5.3 of Ref. [1] and will not be reproduced here.

4. Conclusion

We have analyzed the generation and evolution of spatial irregularities in a quartic potential scalar field inflation model of the very early universe. Since we have used the rapid "reheating" prescription our result for the late time, large-scale, power spectrum of energy density perturbations greatly differs from the result of all previous analyses of this model (which make use of the slow "reheating" prescription). In particular, we find that the power spectrum depends on variables that must be assumed as initial conditions for this approximate model. (It is, at present, not clear if the background field initial conditions of Refs. [17,7,2] are valid for a model with a preinflation epoch; in any case, they do not determine all the needed initial values.) This should be contrasted with the results of our analysis of the exponential potential scalar field model, [1], where the nonlinear potential results in a special solution for the background fields (a time-dependent fixed point, [3]) and ensures that the expressions for spatial irregularities are independent of the spatially homogeneous scalar field initial values.

Although we have not completely succeeded in comparing the quartic potential inflation model to the observations, we have shown, in Ref. [1], that observational data does not unduly constrain the parameters of the exponential potential inflation model (even in the limit when it describes a de Sitter inflation model). We conclude that there is no model-independent evidence that observational data forces inflation epoch parameters to assume exceedingly small values. We again stress two points about the results of our analysis of the quartic potential model: the dependence of the power spectrum on ϵ, eqn. (3.56), only allows for an observational lower limit on ϵ, which agrees with the conclusions of Refs. [1,16] and differs from that of all earlier analyses; furthermore, given preferred values of H_d (or $z_{R\Phi}$) and ϵ, one may always find a value of T such that this model does not obviously disagree with large-scale observational data.

We emphasize that we have had to make a number of approximations in order to reduce the problem to one that is tractable. Although these approximations seem to be physically reasonable (a more detailed discussion can be found in Refs. [1,4,13]) it is useful to bear in mind that our results must be taken as tentative till the approximations are quantitatively justified. In particular, it would be useful to attempt to quantify the validity of the single "fluid" instantaneous transition model and to find a simple model of reheating that may be used to begin a quantitative analysis of this important part of the inflation scenario.

To compare this GUT inspired model of inflation to observational data will require a theoretical estimate of T. One might conceive of analyzing the GUT model in a manner similar to what we have done for the quartic potential model. (In the GUT model, besides the Φ^4 term there is an additional $\Phi^4 \ln\Phi$ term in the potential.) Although this analysis seems to be technically feasible, it is, at present, not clear if it would be of much use, since it seems that one would need to know the appropriate initial conditions for the solutions of the GUT model. It is sometimes suggested that the appropriate initial conditions are those corresponding to initial thermodynamic equilibrium. An investigation of the GUT model using these initial conditions would seem to be a worthwhile exercise (we do not know if this feasible with the currently available tools — it would seem to require estimating the value of the scalar field when tunnelling through a decreasing potential barrier is completed), although one will then be left with the problem of explaining how the preinflation universe ended up in thermodynamic equilibrium. It seems that one is forced to deal with this issue in most models of inflation proposed to date; as far as we are aware, the only exceptions are the exponential potential scalar field models.

In any case, it is perhaps more useful to attempt to begin focussing on a more model-independent, cosmologically-motivated phenomenological, characterization of inflation, since it is quite likely that some of the large time predictions of inflation will soon be subjected to observational scrutiny and one would like to be able to use the results to constrain the short distance physics underlying inflation on the basis of something firmer than collective circumstantial evidence.

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Appendix: Dependence of Background Fields on Choice of Initial Conditions

In this Appendix we show that the final results are not qualitatively sensitive to the values of Φ_{b0i} and $\dot{\Phi}_{b0i}$ (under appropriate conditions). We suppress the subscript b in what follows.

From eqns. (2.5) and (2.6) we find

$$a_0{}^3(t) = B_1 e^{3H_d(t-t_i)} - B_2 e^{-3H_d(t-t_i)}, \tag{A1}$$

where $H_d = (\Lambda/6)^{1/2}$ and

$$B_1 = \frac{1}{2}\left[a_{0i}{}^3 + \left(a_{0i}{}^6 + \frac{c_0{}^2}{2\Lambda}\right)^{1/2}\right], \tag{A2}$$

$$B_2 = \frac{c_0{}^2}{4\Lambda}\left[a_{0i}{}^3 + \left(a_{0i}{}^6 + \frac{c_0{}^2}{2\Lambda}\right)^{1/2}\right]^{-1}. \tag{A3}$$

Using eqn. $(A1)$, eqn. (2.6) may be integrated to give

$$\Phi_0(t) = \Phi_{0i} + \frac{2}{\sqrt{3}}\ln\left[\left(\frac{1+Q}{1-Q}\right)\left(\frac{1-Q\ e^{-3H_d(t-t_i)}}{1+Qe^{-3H_d(t-t_i)}}\right)\right], \tag{A4}$$

where Φ_{0i} is the value of Φ_0 at $t = t_i$ and we have defined

$$Q^2 = \frac{{c_0}^2}{2\Lambda}\left[{a_{0i}}^3 + \left({a_{0i}}^6 + \frac{{c_0}^2}{2\Lambda}\right)^{1/2}\right]^{-2}. \tag{A5}$$

To determine the order ϵ corrections to eqns. $(A1)$ and $(A4)$ we write

$$\Phi(t) = \Phi_0(t) + \epsilon\Phi_1(t), \tag{A6}$$

$$a(t) = a_0(t)[1 + \epsilon f(t)], \tag{A7}$$

and expand eqns. (2.4) and (2.5) to first order in ϵ. We find

$$\ddot{\Phi}_1 + 3\,\frac{\dot{a}_0}{a_0}\,\dot{\Phi}_1 + 3\dot{\Phi}_0\dot{f} - \Lambda\,(\Phi_0 - \bar{\Phi})^3 = 0, \tag{A8}$$

$$2\,\frac{\dot{a}_0}{a_0}\,\dot{f} - \frac{1}{6}\dot{\Phi}_0\dot{\Phi}_1 + \frac{1}{24}\,\Lambda\,(\Phi_0 - \bar{\Phi})^4 = 0. \tag{A9}$$

Using eqn. $(A9)$ to eliminate $\dot{f}$ from eqn. $(A8)$, we may derive an equation of motion for Φ_1,

$$\ddot{\Phi}_1 - \tilde{g}(t)\dot{\Phi}_1 - \tilde{f}(t) = 0, \tag{A10}$$

where we have defined

$$\tilde{g}(t) = -3H_d\,\frac{\left[1 + 6Q^2\ e^{-6H_d(t-t_i)} + Q^4\ e^{-12H_d(t-t_i)}\right]}{\left[1 - Q^4\ e^{-12H_d(t-t_i)}\right]}, \tag{A11}$$

$$\begin{aligned}\tilde{f}(t) =& \Lambda\left[T + \frac{2}{\sqrt{3}}\ln\left(\frac{1 - Q\ e^{-3H_d(t-t_i)}}{1 + Q\ e^{-3H_d(t-t_i)}}\right)\right]^3 \\ & \times\left[1 + \frac{\sqrt{3}}{4}\frac{Q\ e^{-3H_d(t-t_i)}}{\left\{1 + Q^2\ e^{-6H_d(t-t_i)}\right\}}\left\{T + \frac{2}{\sqrt{3}}\ln\left(\frac{1 - Q\ e^{-3H_d(t-t_i)}}{1 + Q\ e^{-3H_d(t-t_i)}}\right)\right\}\right],\end{aligned} \tag{A12}$$

and the parameter

$$T = \Phi_{0i} - \bar{\Phi} + \frac{2}{\sqrt{3}}\ln\left(\frac{1 + Q}{1 - Q}\right). \tag{A13}$$

Changing the independent variable from time (t) to x where

$$x = Q^2\ e^{-6H_d(t-t_i)}, \tag{A14}$$

it is straightforward to establish that the first integral of ($A10$) is

$$\Phi_1'(x) = \frac{(1+x)}{x^{1/2}(1-x)^2}\left[c + \int^x \bar{f}(x')\,dx'\right], \tag{A15}$$

where c is a constant of integration and

$$\bar{f}(x) = \frac{1}{6}\frac{(1-x)^2}{x^{3/2}(1+x)}\left[T + \frac{2}{\sqrt{3}}\ln\left(\frac{1-x^{1/2}}{1+x^{1/2}}\right)\right]^3 \times \left[1 + \frac{\sqrt{3}}{4}\frac{x^{1/2}}{(1+x)}\left\{T + \frac{2}{\sqrt{3}}\ln\left(\frac{1-x^{1/2}}{1+x^{1/2}}\right)\right\}\right]. \tag{A16}$$

When $x \ll 1$, we find [if $T \neq 0$, eqn. ($A13$)],

$$\Phi_1'(x) \simeq -\frac{T^3}{3}\left[\frac{1}{x} + \sqrt{3}\left(\frac{2}{T} - \frac{T}{8}\right)\frac{\ln x}{x^{1/2}} - \frac{3}{T^3}\frac{c}{x^{1/2}} + \cdots\right], \tag{A17}$$

and if $T = 0$,

$$\Phi_1'(x) \simeq \frac{c}{x^{1/2}} + \left(3c - \frac{32}{9\sqrt{3}}\right)x^{1/2} + \left(5c - \frac{16}{3\sqrt{3}}\right)x^{3/2} + \cdots; \tag{A18}$$

it may be verified that these expressions satisfy eqn. ($A10$). Changing back to the original independent variable, we have, in the limit $x \to 0$, if $T \neq 0$,

$$\dot{\Phi}_1(t) = 2H_d T^3, \tag{A19}$$

and if $T = 0$,

$$\dot{\Phi}_1(t) = 0, \tag{A20}$$

where the differences from these values mostly vanish exponentially in time.

Now the $x \ll 1$ limit may be rewritten as

$$\frac{1}{3H_d}\ln Q \ll t - t_i, \tag{A21}$$

which may be interpreted as either a large time limit ($t \gg t_i$) or as a small $Q(\ll 1)$ limit. In this limit we have, to order ϵ,

$$\dot{\Phi}(t) = \frac{c_0}{{a_0}^3} + \epsilon\,[2H_d T^3 + \cdots], \tag{A22}$$

where the ellipses denote terms that decay exponentially in time (some with prefactors that might grow as a power of time); also, in this limit ${a_0}^3 \propto \exp[3H_d(t-t_i)]$, eqn. $(A1)$. So, if T does not vanish, the qualitative dynamical behaviour of $\dot{\Phi}$ is rapidly determined by the order ϵ term in this expression. From similar considerations, it may be shown that, in this limit, the dominant order ϵ term in the scale factor is $\dot{f} \simeq -H_d T^4/8$.

We note that the potential, eqn (1.4), in the range of Φ of interest, describes an unstable dynamical system — Φ must always increase with time (i.e., it will always roll down the hill for reasonable initial conditions). It is instructive to examine the homogeneous scalar field initial conditions in somewhat more detail. These conditions are conveniently classified according to whether $T = 0$ or not. An example of the first set, which we call "quantum fluctuation" initiated rolling is, $\Phi_{0i} = \bar{\Phi}$, $\dot{\Phi}_{0i} = 0$; examples of the second class include $\Phi_{0i} \neq \bar{\Phi}$, $\dot{\Phi}_{0i} = 0$ (potential energy initiated rolling) and $\Phi_{0i} = \bar{\Phi}$, $\dot{\Phi}_{0i} \neq 0$ (kinetic energy initiated rolling). Quantum fluctuation initiated rolling cannot be treated in our approximation because we find, at large time, $\dot{\Phi}(t) = 0$ (which is not the physically correct behaviour), since we have not included the backreaction of the nonzero momentum modes on the homogeneous background fields (it is an interesting exercise to develop a formalism to treat this case; we suspect that the results will not be greatly different from the other cases).

Ignoring the $T = 0$ case, our analysis shows that $\dot{\Phi}(t)$ will be dominated by the second term in eqn. $(A22)$, i.e., for given values of $\dot{\Phi}_{0i}$, t_i and ϵ (which are free parameters in the context of this model), if one waits long enough, the second term dominates. Unfortunately, even though the large time dynamics of the background fields is qualitatively insensitive to the specific value of $\dot{\Phi}_{0i}$, it seems to be quantitatively sensitive to it, because of the dependence of T on $\dot{\Phi}_{0i}$ [see eqns. $(A5)$ and $(A13)$].

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18. By an "isocurvature" perturbation we mean an inhomogeneity which, on large scales, does not perturb spatial curvature. In a model where the stress tensor is dominated by only one

type of matter such a solution usually oscillates inside the Hubble radius and does perturb the curvature of spatial hypersurfaces on small scales.

19. To verify eqn. (3.15) we need to use eqn. $(C2)$ of Ref. [5]. It was pointed out in a footnote in Ref. [1] that the definition of the function F_1 given below eqn. $(C2)$ is incorrect. We note that $(h^{(R)})' = -(3/2)\, c_2^{(R)}\, e^{-x}[1 + (1/2)\lambda_R^2\, e^x] + e^{3x/4}(c_+^{(R)}\bar{W}_1(x) + c_-^{(R)}\bar{W}_2(x))$, where $\bar{W}_1(x) = -6(2/\pi)^{1/2}\lambda_R^{-7/2}\, e^{-7x/4}(i + \lambda_R\, e^{x/2})\exp\left(i\lambda_R\, e^{x/2}\right)$ and $\bar{W}_2 = (\bar{W}_1)^*$.