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Some Observational Constraints on Scalar Field Inflation Models of the Very Early Universe

Bharat Ratra

Joseph Henry Laboratories, Department of Physics,
Princeton University, Princeton, N.J. 08544

ABSTRACT

We show, in a simple class of inflation models of the very early universe, that the large time observational bound on the magnitude of large-scale spatial irregularities does not unduly constrain the parameters of the inflation model. Previous analyses of spatial irregularities in a related exponential expansion inflation model seem to be incomplete; a slightly more complete estimate shows that it is premature to assert that the model has a "fine-tuning" problem.

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The gravitational characteristics of inflation ensure that microphysical length scales in the very early universe are enormously expanded to macrophysical length scales that are now of cosmological significance. Gravitation also ensures that small-scale zero point fluctuations during inflation evolve to become large-scale irregularities in an otherwise spatially homogeneous large time cosmological model. Since these irregularities might be responsible for the observed large-scale structure of the universe, it is of interest to understand their evolution.

Here we highlight two results of an analysis of the evolution of spatial irregularities in a simple class of inflation modified hot big bang models, [1]: observational data does not unduly constrain the parameters of these inflation models of the very early universe; and, for these models the observations suggest an upper bound on the energy density at the end of inflation $\sim 10^{16}$ GeV (they also suggest a lower bound $\sim 10^{7}$ GeV). The second result, if assumed to be not significantly model dependent, points to the following conclusions: primordial supersymmetric or electroweak scale models of inflation are probably inconsistent with observational data; inflation is not necessarily associated with GUT scale physics (although models which stopped inflating late will probably require a new baryosynthesis mechanism); and, the upper limit of $\sim 10^{16}$ GeV (which is well below the Planck scale) validates the semi-classical approximation used in the analysis of Ref. [1] (the use of the, now standard, quantum mechanical initial conditions of Refs. [2,3] in the analysis of Ref. [1] is justified in Ref. [4]).

The first result, that the observational data does not require the parameters of the inflation model to assume extreme values, differs from the conclusions of almost all previous analyses of large time energy density irregularities arising from de Sitter inflation, [5-8]. We believe these previous analyses are incomplete and present a more complete estimate of energy density irregularities in the model used in these references to show that their conclusions are, at the very least, premature.

We would like to determine the large time form of large-scale spatial irregularities which characterize inflation modified hot big bang models of the universe. We model the evolution of a spatially flat universe by dividing it into three epochs: an early scalar field dominated inflation epoch, during which the expansion of the universe is driven by a scalar field (Φ) with an exponential potential, $V(\Phi) \propto \exp(-\Phi/\sqrt{p})$, $p > 1$, which results in a scale factor $a(t) \propto t^p$ (see Sec. 3.1 of Ref. [9]); and conventional radiation dominated and baryon dominated ($\Omega_{\rm now} = 1$) epochs. (We shall constrain the parameters of the inflation model by comparing large time expressions for spatial irregularities to observational data.) In each epoch we only account for perturbations in the dominant form of matter. Since we want to determine the form of large time large-scale "observables" (e.g., the power spectrum of energy density irregularities in the baryon dominated epoch), we need only study a macrophysical model of the universe, i.e., we neglect the microphysical coupling of the scalar field to "radiation" in the early universe and the microphysical coupling of radiation to baryons in the late universe (the main effect of the radiation–baryon coupling is to modify the small-scale form of the large time power spectrum, [10]). Since the details of the reheating transition (the scalar field–radiation transition) are governed by microphysics, they also cannot significantly influence the large-scale form of large time inhomogeneities. The simplest, not totally unrealistic, approximation that one can make is to treat the transitions as instantaneous, i.e., require that the equation of state (and hence the pressure) change instantaneously and discontinuously at the spatial hypersurface on which the local (total) energy density reaches an appropriate value (the transitions are governed by local physics and this prescription will have to be verified by a microphysical calculation). We emphasize that although it is not incorrect to take the macrophysical scalar field potential to be the same as the microphysical potential during inflation (i.e., when the scalar field is "on top of the hill") there is no reason to believe that the collective effects of the small-scale physics of reheating can be modelled, even on small scales, by the same spatially homogeneous mode of the scalar field rolling down the steep part of the microphysical potential, as is used "on top of the hill". A more detailed description of the model is given in Ref. [1].

To derive expressions for late time large-scale deviations from spatial homogeneity we proceed as follows: The equations of motion that govern the evolution of irregularities are the linearized equations of covariant conservation of stress-energy and the linearized Einstein equations (we work in synchronous gauge). Using the techniques developed in Ref. [11] we derive closed-form solutions of these equations in each of the three epochs. To determine the constants of integration in the inflationary epoch expressions we use the quantum mechanical initial conditions of Ref. [2]; to determine the constants of integration in the large time expressions for irregularities we derive joining conditions at the transitions by requiring that the equations of motion not become singular on the transition spatial hypersurfaces. Since our model does not correctly describe the relevant microphysics we focus on those perturbations that first reentered the horizon in the baryon dominated epoch, i.e., we approximate our expressions to ensure that we only retain those spatial momentum modes that were outside the horizon during the reheating and radiation–baryon transitions. Further details may be found in Ref. [1].

One result of interest is that in the limit $q = 2\epsilon^2 \sim 0$ (where $q = 2/p$) the inflation epoch scalar field potential reduces to the step-function potential of Ref. [3],

$$V(\Phi) = 2\Lambda[1 - \epsilon\,(\Phi - \Phi^{(0)})], \tag{1}$$

where Λ is the cosmological constant during inflation and $\Phi^{(0)}$ is a constant — this limit corresponds to exponential expansion inflation. In this limit the large time power spectrum of large-scale, gauge-invariant adiabatic energy density irregularities,

$$P(k,t) \equiv \langle \delta_B(k,t)\delta_B(-k,t)\rangle = \frac{16\pi}{m_p^2}\, k^{2(1-\nu)}|A|^2 a^2 \tag{2}$$

(here δ_B is the fractional baryonic energy density perturbation, k is the coordinate spatial momentum, $\nu = (2+q)/[2(2-q)]$, m_p is the Planck mass and $|A|$ is a numerical factor that depends on the time of reheating and the time of the radiation–baryon transition and is defined in Sec. 5.2 of Ref. [1]), reduces to the scale-invariant form

$$P(k,t) \propto \frac{1}{\epsilon^2}\, ka^2, \tag{3}$$

where the numerical prefactor increases as the potential, (1), becomes flatter and as the model approaches exact de Sitter inflation. The geometrical reason for this behavior is discussed in Sec. 5.2 of Ref. [1]; we note that the linear perturbation analysis breaks down if ϵ is exceedingly small.

We have not yet determined the observationally desired range of the two free parameters of the model: the redshift of reheating and q, the index of the scalar field potential. Since our analysis is only valid on large scales, we fix the normalization of the power spectrum by requiring that the theoretical expression for the large-scale peculiar velocity (in the baryon dominated epoch) agree with what is observationally found. This results in a relation between q and the redshift of reheating and the model is then completely specified by the value of q. (For further details see Sec. 5.4 of Ref. [1].) We note that this one parameter family of inflation models is not obviously inconsistent with observational data for a fairly large range of q; furthermore, no parameter assumes an exceedingly small value ($\lesssim 10^{-10}$) that previous analyses, Refs. [5-8], conclude is observationally desired.

Previous results for irregularities arising from de Sitter inflation do not agree with our expression. Besides the disagreement over whether observational data requires that inflation parameters assume extreme values, our expression for the large-scale, baryon dominated epoch, power spectrum depends on the redshift of reheating and the redshift of the radiation–baryon transition in a manner which differs from earlier results, [5-8]. Before examining a possible reason for these differences we note that Refs. [5-7] ignore metric perturbations during inflation (which might not be too crude an approximation for exact de Sitter inflation, but is incorrect for almost any other kind of inflation, Sec. 3.1 of Ref. [1]); we emphasize that metric perturbations cannot be ignored in the radiation and matter dominated epochs (Sec. 2 of Ref. [11]) and that the joining conditions for spatial irregularities at the transitions mix energy density, peculiar velocity and metric perturbations (Sec. 4.2 of Ref. [1]) — it is clear that neglecting metric perturbations will result in a quantitatively incorrect estimate although sufficient caution should ensure that the final result is not qualitatively incorrect. We also note that the joining conditions used in Ref. [8] disagree with the joining conditions we have derived by requiring that the equations of motion not become singular at the transition spatial hypersurface (see Sec. 4.3 of Ref. [1]).

We believe that the final results of earlier analyses are incorrect and that there does not yet seem to be any evidence that observational data unreasonably constrains the parameters of the inflation model. In what follows we focus on the de Sitter inflation model studied in the first of Refs. [5] (similar results have been found in Refs. [6-8]; for further details see Sec. 5.2 of Ref. [1]). The scalar field potential is taken to be of the form $V(\Phi) \simeq V_0 - \lambda\Phi^4/4$, where the constant, Φ independent, term V_0 is large and dominates during inflation (it determines the value of the constant Hubble parameter during inflation), the second small term causes the scalar field to slowly roll "on top of the hill" and λ is a coupling constant which governs the deviation away from exact de Sitter inflation [this potential should be compared to the exponential expansion limit form of the potential of the model we have examined, (1)]. We note that the $\lambda\Phi^4$ potential is meant to be an approximation to a more complicated GUT potential, however, as with previous analyses the following remarks hold only for a model with the above potential — it is possible that an analysis of the GUT model which motivated the $\lambda\Phi^4$ approximation will lead to a quantitatively different answer. [For the purpose of observationally constraining the parameters of the inflation model, both the GUT model potential and the $\lambda\Phi^4$ potential have two free parameters. For the GUT model these are the gauge coupling constant, g, and the expectation value of the scalar field at the minimum of the potential, σ; for the approximate potential the parameters are $V_0 \propto (g\sigma)^4$ and $\lambda \propto g^4$.

The "fine-tuning" problem of Refs. [5-8] is the following: For a fixed GUT scale, i.e., for a fixed value of $g\sigma$ (and hence a fixed Hubble parameter during inflation), the observations seem to constrain λ (and hence g during inflation) to a numerical value that, when used in the renormalization group equations to determine the magnitude of coupling constants at low energies, is inconsistent with the numerical value expected from "high" energy experimental data (this is not the most general "fine-tuning" problem that these models might have).]

In our notation the expression for the power spectrum at second horizon crossing found in previous analyses is

$$\hat{P}(k) \propto \frac{H^2}{\dot{\Phi}_0^2}\, k^3 \left\langle \phi(k)\phi(-k) \right\rangle \tag{4}$$

where H is the Hubble parameter, Φ_0 is the spatially homogeneous background part and ϕ the small spatially inhomogeneous part of the scalar field, a dot denotes a derivative with respect to time and the hat distinguishes this expression for the power spectrum from our expression; all time dependent quantities on the right hand side of this equation are evaluated at the time when the mode first crosses the horizon. Consistent with above comments on earlier analyses we believe this expression is qualitatively reasonable but probably not quantitatively correct. If one only retains the gauge invariant adiabatic solution in the expression for ϕ "on top of the hill" one finds $k^3 \left\langle \phi(k)\phi(-k) \right\rangle \propto H^2$ (which we believe is consistent with what was found earlier); since H is both time and λ independent "on top of the hill" in the de Sitter inflation model (it is determined by V_0; H is likely to be weakly dependent on both t and λ in the underlying GUT model), the t and λ dependence of the power spectrum is $\hat{P}(k) \propto 1/\dot{\Phi}_0^2$. These references consider a solution of the form

$$\Phi_0(t) = \left(\frac{3H}{\lambda}\right)^{1/2} (-t + t_*)^{-1/2} \tag{5}$$

for the homogeneous scalar field, where $t < t_*$ and the constant of integration t_* is claimed to be the "time of reheating" (for this solution and for a suitably small $t < t_*$ the $\ddot{\Phi}_0$ term in the scalar field equation of motion is insignificant). We note that $\dot{\Phi}_0(t) = (3H/\lambda)^{1/2}(-t +$

$t_*)^{-3/2}/2$, which implies $\hat{P}(k) \propto \lambda$, i.e., the power spectrum decreases as the model approaches exact de Sitter inflation — in direct contradiction to what we have found. The constant of integration in this solution of the first order (in time) differential equation was determined by the final condition $\Phi_0(t_*) = \infty$. This procedure is incorrect since neither is the general solution of the differential equation, used by these authors, mathematically valid nor is the approximate physical model correct when $\Phi_0 = \infty$. Ignoring for a moment that it is not conventional to use a final condition to determine a constant of integration in a solution of a time-evolution equation, we digress to consider the final condition $\Phi_0(t_*) = \Phi_{0*}$, then $\Phi_0(t) = [{\Phi_{0*}}^{-2} + (\lambda/3H)(-t+t_*)]^{-1/2}$, which reproduces (5) when $\Phi_{0*} = \infty$, and

$$\dot{\Phi}_0(t) = \frac{\lambda}{6H}\left[\frac{1}{{\Phi_{0*}}^2} + \frac{\lambda}{3H}(-t+t_*)\right]^{-3/2}; \tag{6}$$

we note that for sufficiently small λ the slow rolldown approximation is valid. If $\Phi_{0*} \neq \infty$ then as $\lambda \to 0$, $\dot{\Phi}_0(t) \to 0$ [and $\hat{P}(k) \propto \lambda^{-2}$], while for (5), $\dot{\Phi}_0(t) \to \infty$ in this limit. In any case, this solution is also not correct, both because of the final condition and because the potential has not been treated correctly. Since the small $\lambda\Phi^4$ term is only an approximation to a more complete potential, the relevant growing solution on top of the hill is

$$\Phi_0(t) = c_0 + \lambda\frac{c_0^3}{6H}t, \tag{7}$$

where c_0 is a constant of integration and we have worked to first order in λ. We note that the time dependence of this solution agrees with (5) in the limit $t \ll t_*$. With this solution, $\hat{P}(k) \propto \lambda^{-2}$, which increases as the model approaches exact de Sitter inflation. Since current observations only provide an upper bound on $\hat{P}(k)$, they can only result in a lower bound on λ. To determine whether this lower bound clashes with the desired microphysical range of λ will require a detailed analysis of the evolution of perturbations in this model, similar to that presented in Ref. [1]. [We emphasize that we have only shown that if one uses (7) instead of (5) the power spectrum scales with λ in a manner which differs from what was found earlier.]

The analysis of the Dimopoulos-Raby supersymmetric GUT model presented in Sec. III.B of Ref. [8] also illustrates the effect of the de Sitter inflation limit divergence. In this model the power spectrum $\hat{P}(k) \propto (c_2)^{-2}$ [c_2 is the parameter which governs the deviation of the scalar field potential from that corresponding to a constant cosmological constant] which increases as the deviation from exact de Sitter inflation decrease.

Although the simple-minded approximation to the reheating transition used in Ref. [1] leaves open the possibility that a more accurate treatment might quantitatively change the de Sitter limit conclusions, we suspect that the relevant small-scale physics (which operates on microphysical time scales) will not significantly influence the large-scale form of the power spectrum. It would, however, be difficult to overemphasize the need for a semi-realistic examination of the reheating subscenario of the canonical inflation scheme.

In conclusion, we have studied the evolution of spatial irregularities in a simple class of inflation models of the very early universe with a view towards beginning a quantitative comparison of inflation theory to observational data, which we hope to return to elsewhere. The models of Ref. [1] include the usual exponential expansion inflation model, however, the results found there for this model do not agree with those found earlier for similar models. We have shown that previous estimates are not quite complete and that there does not yet seem to be any reason to believe that observational data unduly constrains the parameters of inflation models of the very early universe.

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References

1. B. Ratra, Princeton preprint PUPT-1102 (1989).

2. B. Ratra, Phys. Rev. **D31**, 1931 (1985).

3. W. Fischler, B. Ratra and L. Susskind, Nucl. Phys. **B259**, 730 (1985).

4. B. Ratra, Princeton preprint, PUPT-1103 (1989), Phys. Rev. **D** (1989) (in the press).

5. A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982), Phys. Rev. **D32**, 1899 (1985).

6. S. W. Hawking, Phys. Lett. **115B**, 295 (1982).

7. A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).

8. J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. **D28**, 679 (1983).

9. B. Ratra and P. J. E. Peebles, Phys. Rev. **D37**, 3406 (1988).

10. P. J. E. Peebles, Ap. J. Lett. **263**, L1 (1982).

11. B. Ratra, Phys. Rev. **D38**, 2399 (1988).