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Inflation Generated Cosmological Magnetic Field

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ABSTRACT

The cosmological consequences of adding a coupling, between the scalar field Φ responsible for inflation and an abelian gauge field (A_μ), $\propto e^{\alpha\Phi}F_{\mu\nu}F^{\mu\nu}$ ($F_{\mu\nu}$ is the field-strength tensor of A_μ and α is a parameter), to the exponential potential scalar field inflation model [with potential $\propto \exp(-\Phi/\sqrt{p})$, $p > 1$], are investigated. The model is divided into three epochs: scalar field, radiation, and baryon dominance. In each epoch we derive the synchronous and temporal gauge linear perturbation equations of motion. To linear order A_μ does not influence the evolution of the gravitational, scalar field, and fluid parts of the model. Ignoring charged particles during inflation, we present a general closed-form expression for the inflation epoch gauge field and determine the constants of integration from quantum mechanical initial conditions. At the inflation-radiation transition we assume that the potential for Φ is such that we may approximate the new coupling by the standard gauge field term $\propto F_{\mu\nu}F^{\mu\nu}$ during the radiation and baryon epochs. Accounting for charged particles during radiation and baryon dominance by using a simple Ohm's law, we present a general closed-form solution for A_μ during radiation dominance, derive and use joining conditions at the inflation-radiation transition to determine the constants of integration, and show that in the large-conductivity limit the electric field is suppressed. We then solve the baryon epoch large-conductivity equations of motion, derive and use joining conditions at the radiation-baryon transition to determine the constants of integration, and evaluate the baryon epoch magnetic field two-point function. Requiring that the (smoothed) Fourier transform of this two-point function be infrared finite and that the inflation epoch large-scale gauge field energy density be significantly smaller than that of the homogeneous scalar field, we find that the spatial momentum k dependence of this two-point function (and magnetic energy density) is $k^{\hat{n}}$, $-3 < \hat{n} < 2$. On a scale of one-thousandth the present Hubble scale the present magnetic field strength ranges between $\lesssim 10^{-65}$G and $\gtrsim 10^{-10}$G. The larger values (corresponding to $\hat{n} \sim -3$) are for models close to the de Sitter inflation model, with a moderately large value of α (~ 20), and provide an example of a primeval magnetic field that might be amplified by galactic collapse (with or without amplification by galactic differential rotation) to produce the observed galactic magnetic fields, while smaller values might provide an initial field needed for possible amplification by galactic dynamos.

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1. Introduction and Summary

There is convincing observational evidence for an intragalactic disk magnetic field, which has a small-scale ($\sim 10^2$pc) random component and a large-scale component uniform on the few kpc scale, both with field strength of order a few μG (microgauss), [1-8]. There is fairly convincing evidence for a large-scale field of similar magnitude and coherence length in other galaxies, [4,5,9,6,10,11]. There is observational evidence for a field of a few μG, ordered on the few kpc scale, in galactic halos, [4,10,12]. There is evidence for low-redshift intracluster magnetic fields of μG strength, possibly ordered on the few $\times$ 10kpc scale (small compared to the cluster size), [9,13,10,14,7] (for the situation on even larger scales see Refs. [15]). There are suggestions that there might be large-scale μG fields at larger redshifts ($\lesssim 3$) which might be associated with large galactic halos (or galaxy groups) (Refs. [9,16,17]), or with large protogalactic disks (Refs. [18,17,19,20]). We emphasize that in most of these systems there is also evidence for smaller coherence-length magnetic fields.

Although the strength and coherence length of large-scale magnetic fields is not accurately known (the statistics are not great and magnetic fields are usually not directly measured and to extract them from observational data one needs the not yet accurately determined density of charged particles), observational evidence for pervasive μG fields, uniform on the $\lesssim 10$ kpc scale, is persuasive. The situation on larger scales is still quite unclear, and it has not yet been decided if there is a large-scale intergalactic field, although it does seem to have been shown that such a field must be weaker than $\sim 10^{-3}\mu$G, [3,9,10].

Given the densities and temperatures of these gaseous systems, the setting for a theoretical study of the evolution of galactic scale magnetic fields is the theory of magnetohydrodynamics (MHD). The MHD equations of motion are the hydrodynamic equations for an electrically conducting fluid and the Maxwell electromagnetic equations, with proper account being taken of the interaction between the conducting fluid and the electromagnetic field (and with the displacement current neglected), Ch. 1 of Ref. [21] and Secs. 36 – 38 of Ref. [22]. Having neglected the displacement current it may be shown that the electromagnetic equations of motion are summarized by a first order in time equation for the magnetic field (the induction or hydromagnetic equation, Sec. 4.1 of Ref. [1]). All terms in this equation are linear in the magnetic field (there is no source term). This means that if the initial field vanished the field at any later time would also vanish (in the MHD approximation).

It is, therefore, convenient to divide the study of galactic magnetic fields into two parts: the production of an initial 'seed' field (by a 'non-MHD' process); and the subsequent MHD evolution of this field. It is usually assumed that the initial seed field is much smaller than the present galactic magnetic field, and so interest is focussed on those solutions of the MHD equations which amplify an initial field. As in most other situations in astrophysics, there are two classes of galactic seed field models: the 'bottom-up' models in which the seed field is produced on scales much smaller than the galactic scale (e.g., in stars); and, the 'top-down' models in which the field is produced on scales much larger than the galactic scale (on cosmological scales in the early universe).

1.1 Large Time Magnetic Field Amplification Mechanisms

In this section we briefly discuss the two most studied processes for the amplification of a large-scale seed magnetic field. We note that these are phenomenological models, i.e., given the need for a quantitative accounting of turbulent processes it is not unexpected that neither model has yet been rigorously derived from the underlying theory.

The fluid in a collapsing protogalaxy is quite likely turbulent. The evolution of a magnetic field in such a system is governed by the MHD equations for an electrically conducting turbulent fluid and a magnetic field. (For the purposes of this qualitative discussion the presence of gravitational and other fields may be ignored.) The complete MHD equations for such a system are neither amenable to analytical study nor to numerical analysis. As a result attention has been focussed on idealized models. In a first approximation one examines kinematical models, i.e., models in which the hydrodynamic equations are ignored, a plausible turbulent velocity field is assumed, and attention is focussed on the hydromagnetic equation. In this approximation it is not possible to analyze the effect of the magnetic field on the fluid motion. We note that the energy density of a typical galactic magnetic field is of the order of the energy density in a typical galactic turbulent velocity field.

Even with the kinematic simplification there does not yet seem to be a rigorous method for deriving the 'effective' hydromagnetic equation for the mean magnetic field (at least not without further simplifying assumptions that are probably not valid in the systems of interest). It has been argued that 'integrating out' the random part of the fluid velocity and magnetic fields results in an 'effective' hydromagnetic equation for the mean magnetic field that differs from the 'bare' hydromagnetic equation, Ref. [23] (especially Chs. 7 and 8), Ref. [1] (especially Chs. 17 and 18), Ref. [24], and references therein. To lowest order and for non-mirror-symmetric (but spatially homogeneous and isotropic) turbulence the new terms in the mean-field hydromagnetic equation are: an electric field proportional to the mean magnetic field (the α effect); and an addition to the coefficient of Ohmic diffusion (the β effect). The coefficients of these terms, α and β, are different combinations of correlation functions of the turbulent velocity field (in lowest order α vanishes for mirror-symmetric turbulence). We note that a positive β can be interpreted as leading to a reduced 'effective' electrical conductivity or an increased 'effective' magnetic diffusivity, so that even when the Ohmic diffusion coefficient vanishes (perfect 'ordinary' conductivity and 'frozen in' magnetic field lines in a non-turbulent electrically conducting fluid) the β effect (or turbulent diffusion) allows magnetic field lines to diffuse through the electrically conducting turbulent fluid.

Attempts to systematically extend this derivation (in the case of interest, when the turbulent eddy correlation time is not infinitesimal) suggest that higher order terms in the perturbation expansion are significant, Refs. [25] and Sec. 18.4 of Ref. [1]. [A summary of earlier work (with extensions) on the evolution of a magnetic field in a turbulent conducting fluid may be found in the paper of Kraichnan and Nagarajan [26], where it is argued that "it seems necessary to treat the detailed dynamics of the turbulence *quantitatively*" and that this "seems beyond the capabilities of the kinds of approximate turbulence theories which are now available"; in spite of intense effort in the two and half decades since this paper was written, this conclusion still seems to hold in the case of interest.]

There seem to be two kinds of higher order terms: terms with a larger number of spatial derivatives than those in the lowest order effective hydromagnetic equation; and terms with the same number of spatial derivatives. Terms with a larger number of spatial gradients may be ignored on sufficiently large spatial scales (provided the mean magnetic field is sufficiently smooth), while those with the same number of spatial derivatives will 'renormalize' the coefficients in the lowest order effective equation (i.e., new combinations of turbulent velocity correlation functions are added to the lowest order expressions for the α and β coefficients; the resulting expressions are not positive definite). For the purpose of a large-scale phenomenological analysis of mean field amplification the lowest order form of the effective hydromagnetic equation should suffice; however, to not prejudge the case the numerical values of the α and β coefficients must be treated as parameters of the model. (Parenthetically we note that the 'new' terms in the effective equation vanish for vanishing magnetic field and so do not vitiate the need for a 'non-MHD' seed field.)

The Ohmic conductivity of the galaxy is large enough to ensure that the Ohmic dissipation timescale for magnetic flux removal is much larger than the age of the universe. One might be inclined to assume that the α and β terms in the effective hydromagnetic equation are negligible (compared to the velocity term). Then, as noted by Hoyle and Ireland [27], galactic differential rotation would drag a weak 'frozen in' primeval magnetic field and linearly (in time) amplify it. They also noted that (in the absence of other processes) this amplification mechanism would soon (after 4 – 5 rotation periods) result in a tightly wound galactic field, contrary to what is observed. Implicitly also neglecting the α and β effects, Piddington [28] (and in passing, also Kardashev [29]) noted that a weak primordial magnetic field would also be amplified during the collapse of a protogalaxy (in the 'frozen' flux limit a weak field would increase as the square of the ratio of initial and final radii during isotropic collapse, with further amplification from anisotropic collapse to a disk), and went on to combine this process with the differential rotation amplification process [30,31].

Kulsrud [32] has pointed out that the galactic magnetic field is mostly anchored to the (rare) ionized component of dense clouds in the disk and, by quantitatively studying a simplified model, has argued that magnetic field lines (that either enter and leave from above the disk or enter and leave from below the disk) will rapidly move away from the disk (while the field is being amplified by differential rotation) and so the 'wrap-up' problem will be alleviated (field lines that enter from above and leave from below, or vice versa, will remain in place). (That rapid vertical diffusion of the magnetic field might phenomenologically help to resolve the 'wrap-up' problem has also be noted in Ref. [33].) Further elaboration of the amplification by collapse and differential rotation model may be found in Refs. [34,35], where it is argued that a steady state might be reached and that the final field might not be very different from the observed galactic magnetic field. A less optimistic discussion of some features of this kind of amplification model may be found in Sec. 17.7 of Ref. [1].

Arguing that the α and β effects cannot be neglected, and only retaining the lowest order terms in the expressions for α and β, Parker [36] and Vainshtein and Ruzmaikin [37] pointed out that these terms along with galactic differential rotation would exponentially (in time) amplify a pre-existing magnetic field (such a process is called a dynamo). If appropriate numerical values are chosen it can be ensured, in the kinematic approximation, that the lowest field modes are amplified (on an exponential doubling timescale of order the galactic rotation period). In their second paper Vainshtein and Ruzmaikin [37] noted that to achieve a steady-state field the exponential amplification must halt. They pointed out that as the field was amplified it must feedback on the fluid motion and it might alter the α and β coefficients before it substantially influenced other motions of the fluid (i.e., before the large-scale magnetic and turbulent velocity energy densities reached equipartition); to study these effects one would need to be able to say something about the solutions of the hydrodynamic equations. Further elaboration of the dynamo scenario (including applications to intragalactic, galactic halo, and intracluster fields) may be found in Refs. [38].

There is a dynamo scenario in which primordial cosmological turbulence is used to amplify a large-scale primeval field before recombination [39]; this is now thought to be not that efficient an amplification process [40] (furthermore the initial conditions required for primeval turbulence are not consistent with the scalar field dominated inflation picture of the very early universe [41,42]). A post-recombination amplification process has also been studied [43]; the required turbulence is ascribed to fluctuations within protogalactic objects.

We have already mentioned one of the difficulties associated with deriving, from first principles, the relevant 'effective' equations needed for a study of the evolution of the galactic field. The effective hydromagnetic equation used to describe dynamo models has been heuristically criticized, Refs. [44] and references therein. (For a discussion of some of these issues see Cowling's review [45].) It has been argued that both the large-scale mean field and the small-scale part of the field are amplified (in a turbulent electrically conducting fluid); dynamo amplification proponents suggest that the scale of the small-scale part of the magnetic field will be decreased to the point where Ohmic dissipation (or a related process) will limit the growth of the small-scale field. An interesting point raised in the above papers is whether this actually happens. A quantitative version of the argument, in the context of the galactic dynamo model, is given in Refs. [34,46]; in these papers Kulsrud suggests that the small-scale fields will only be Ohmically dissipated until the mean magnetic field reaches $\sim 10^{-9}$G (roughly three orders of magnitude below the observed galactic field) and that by this time the small-scale fields would have influenced the fluid turbulence to such an extent that the dynamo shuts down. Support for a related argument is found in the results of the numerical simulation of a two-dimensional model [47].

In summary, there are two plausible classes of galactic magnetic field amplification models: the dynamo model; and the collapse and differential rotation model. Neither model has been rigorously derived from first principles (and both of them have been criticized), nevertheless they are phenomenologically quite valuable.

1.2 Large-Scale 'Seed' Magnetic Field Generation Mechanisms

As mentioned above, both of the field amplification mechanisms require an initial 'seed' field to feed on. We have summarized, at the end of Sec. 6.2 below, the needed magnitude of galactic scale seed fields: the present galactic field might be the result of amplification by isotropic collapse (without amplification by differential rotation) of an intergalactic field of present strength $\sim 10^{-10}$G, or with amplification by anisotropic collapse and differential rotation a present intergalactic field of $\sim 10^{-13}$ – 10^{-12}G is needed; a present intergalactic field of strength $\sim 10^{-22}$ – 10^{-16}G might suffice for the galactic dynamo scenario.

The needed 'seed' might be produced by a 'non-MHD' process that operates on scales much smaller than the galactic scale (e.g., in stars, star clusters, galactic nuclei, etc.), on scales comparable to that of the galaxy, or on scales larger than that of the galaxy (a primordial cosmological field). Stars rotate many more times in their lifetime than galaxies do in their lifetime; as a result an amplification process whose timescale is of the order of the rotation timescale would be more effective in smaller scale objects. It is known that stellar winds and explosions spread magnetic lines of force over distances that can be as large as $\sim 10^2$pc (and are probably responsible for the small-scale component of the galactic field); it seems clear that in the absence of a significant β effect (turbulent diffusion) differential rotation alone cannot use these small-scale fields as a seed to produce a smooth large-scale galactic field. We note that during star formation some mechanism must remove angular momentum from the collapsing protostar (otherwise the final object would be a low density, centrifugally supported, cloud); a magnetic field would help to accomplish this (we emphasize that there are other, less-studied, mechanisms for angular momentum removal) [48].

In what follows we list some of the 'non-MHD' processes that have been invoked to generate a large-scale, precollapse, galactic seed field. (We ignore magnetic monopole related processes.) A variant of at least one of these processes has also been used to generate a smaller scale seed; we do not touch upon the issue of producing small-scale seed fields here. A comprehensive review of seed field generation mechanisms does not seem to exist; some of the processes used are discussed in Refs. [49] (for more recent discussions of some of these and other, smaller scale, processes see Refs. [9,50]). Parenthetically we note that following a remark of Hoyle [51], that the required seed field might be a remnant of an earlier epoch, some effort was devoted to studying cosmological models containing a magnetic field [52].

In the radiation epoch of the hot big bang model Thomson scattering couples the electrons of the ionized plasma to cosmic microwave background photons; in the presence of nonzero transverse peculiar velocity (vorticity) perturbations the ions spin down faster than the electrons [53]. To compensate for the small charge separation an electric field with a nonzero curl (and so a magnetic field with a nonzero time derivative) develops; the resulting present intergalactic field strength is $\sim 10^{-22}$G [53]. (Scalar field dominated inflation models do not generate late time vorticity perturbations in the large-scale linear regime [41,42].)

Phase transitions might produce magnetic fields. It has been suggested that a first-order phase transition might produce an intergalactic field of present strength $\lesssim 10^{-13}$G [54] (this number seems to have been estimated from energy equipartition; it is not obvious that this assumption is justified). The QCD phase transition might generate a field $\sim 2\times 10^{-17}$G on a scale $\sim 5\times 10^{10}$cm at recombination (and so $\sim 2\times 10^{-23}$G on a present scale of $\sim 2\times 10^{-5}$pc or $\sim 6\times 10^{-38}$G on a present scale of $\sim$ 100kpc) [55]. The electroweak phase transition might generate a field of present strength $\sim 10^{-30}$G on a scale of $\sim$ 100kpc [56].

An early epoch of inflation would generate small-scale quantum mechanical magnetic field perturbations (in much the same way as energy density, peculiar velocity, and gravitational wave perturbations are generated, [57,58,41]) which would then be stretched to larger scales by the rapid expansion of the inflation epoch. Adding an abelian gauge field, described by the standard Maxwell action, to the inflation modified hot big bang model one finds a magnetic field of present strength $\sim 10^{-59}$G on a scale of one-thousandth the Hubble scale, [59]. By giving the 'photon' a 'mass' (which is a function of the Ricci scalar and tensor), and so explicitly breaking gauge-invariance, it is possible to get a field of present strength $\sim 10^{-10}$G on a scale of a megaparsec [60]. More conventional gauge-invariant couplings of the 'photon' to spacetime curvature might result in a present field $\sim 10^{-40}$G [60]. We have studied an inflation model in which the 'photon' is gauge-invariantly coupled to the scalar field responsible for powering the inflation expansion; depending on the values of parameters the present field varies between $\lesssim 10^{-65}$G and $\gtrsim 10^{-10}$G (possibly even $\sim 10^{-9}$G) on a scale of one-thousandth the Hubble scale, Ref. [59] and Sec. 6.2 below.

A strong enough primordial magnetic field would have a significant effect on galaxy formation [61] (also see Sec. 17 of Ref. [62] as well as Refs. [31,52,63]). The largest magnetic field strength we have found is a little smaller than that considered in Ref. [61]; it might be of interest to examine the dynamical consequences of such a field. For some range of parameters the magnetic field we find, while too weak to be dynamically important, might still influence diffusion processes in the manner envisaged, for instance, in Ref. [50].

1.3 Exponential Potential Scalar Field Inflation Modified Hot Big Bang Model and Maxwell-Scalar Field Coupling

The evolutionary history of simple inflation modified hot big bang models is conveniently divided into three epochs [41,42,64]: an early inflation epoch (usually dominated by a scalar field with an appropriate potential) and the late time radiation and baryon dominated epochs.

The simplest dependence of the inflation epoch homogeneous scalar field energy density $\rho_{b\Phi}$ on the FLRW scale factor a is $\rho_{b\Phi} \propto a^{-q}$; inflation models require $0 < q < 2$. It is straightforward to establish that a scalar field (Φ) potential $\propto \exp(-\Phi/\sqrt{p})$, $p > 1$ ($p = 2/q$) results in this scale factor dependence of $\rho_{b\Phi}$ [65]. (Related models have been studied in Refs. [66,67].) At the inflation-radiation transition the scalar field potential is allowed to instantaneously drop to zero and the scalar field energy density is assumed to convert to radiation energy density; this provides a simple large-scale phenomenological model of the microphysical reheating transition. (The results which follow on making use of this instantaneous transition model should not be significantly changed as long as the transition timescale is much less than the Hubble time scale at this epoch [66,68,64].)

In a first approximation, in each epoch we ignore spatial inhomogeneities in all but the dominant form of matter. We also assume that the transitions between epochs are instantaneous. In each epoch we can then derive general solutions of the synchronous gauge linear perturbation equations of motion [69]. The constants of integration in the inflation epoch solutions are determined from quantum mechanical initial conditions [70,57,58], while those in the radiation and matter epoch solutions are determined from joining conditions at the transitions [71]. We require joining conditions in these models since we have only accounted for spatial irregularities in the dominant form of matter in each epoch. The joining conditions are derived by requiring that the equations of motion remain singularity-free on the transition spatial hypersurface [71]. In Ref. [41] the transition hypersurface was taken to be that on which the total energy density is spatially homogeneous.

Parenthetically we note that in Ref. [41] we found that observational data does not unduly constrain the parameters of the exponential potential scalar field inflation model. A similar conclusion holds in a number of other models [42,66,72]. In particular, consistency with large-scale observational data does not require that a parameter of the inflation epoch scalar field potential be exceedingly small ($< 10^{-10}$) as had previously been believed to be essential. One reason for this difference is that generic inflation models have at least two free parameters (the slope of the potential during inflation and the height of the potential at the end of inflation) [41,42,66,72], while the quartic potential model, with the slow reheating prescription [64], which was found to be 'fine-tuned' (i.e., needed an exceedingly small scalar field quartic coupling constant during inflation and reheating for observational consistency) is nongeneric (i.e., the late time energy density power spectrum is only weakly, logarithmically, dependent on the height of the scalar field potential at the end of inflation) [72].

An epoch of inflation will stretch vector potential inhomogeneities in much the same manner as energy density and other spatial irregularities are stretched. A $U(1)$ gauge field A_μ could be added to the scalar field inflation model by adding the standard term $\propto e^{-2}\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$ to the Lagrangian density (the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, g is the metric tensor determinant, and e is the Maxwell coupling constant); however the conformal symmetry of the standard Maxwell model ensures that the present large-scale magnetic field is exceedingly small ($\sim 10^{-59}$G on a scale of one-thousandth the present Hubble scale). To get around this obstacle we have chosen instead to include a term of the form $\propto \bar{e}^{-2}\sqrt{-g}e^{\alpha\Phi}F_{\mu\nu}F^{\mu\nu}$ in the inflation epoch Lagrangian density ($\bar{e}$ is the Maxwell coupling constant during inflation). This adds a new free parameter, α, to the exponential potential scalar field inflation model of Ref. [41] (the other two parameters are the redshift of the inflation-radiation transition and the variable p of the inflation epoch scalar field potential).

Once again we assume that the transitions occur instantaneously when the local energy density drops to an appropriate value. We also assume that at the inflation-radiation transition the scalar field potential can be chosen to ensure that the spatially homogeneous part of Φ (Φ_b) develops a constant vacuum expectation value, so that after reheating $\bar{e}/\left\langle \exp(\alpha\Phi_b)\right\rangle^{1/2}$ is the standard Maxwell coupling constant (and to leading order the new term in the Lagrangian density reduces to the standard Maxwell form). We shall also assume that after reheating the scalar field potential has been adjusted to ensure that the model satisfies the large time observational and experimental constraints on such a coupling.

We note that the inflation epoch Maxwell-scalar field coupling term we consider is of the same general form as the dilaton-vector field coupling term present in some superstring models. In the superstring scenario it is believed that once supersymmetry is broken string loop corrections will generate a potential for the dilaton [73], the form of which might ensure that the time dependence of the dilaton vacuum expectation value does not conflict with observational bounds on the time variation of low-energy coupling constants.

To study the effects of the new term we need to extend the techniques we have previously used. To simplify matters we first consider a zero conductivity universe. It is straightforward to derive joining conditions for the Maxwell field at the inflation-radiation transition; with the assumption that Φ develops a constant vacuum expectation value at the reheating transition (and that after reheating deviations from this value may be ignored in a first approximation), we find that the electric and magnetic fields must not jump at this transition.

In a first approximation it is not unreasonable to ignore the effects of charged particles (which couple to the gauge field) during inflation. However, during radiation and baryon dominance these effects must be accounted for. This is accomplished by using a simple Ohm's law to express the electrically conducting fluid current in the curved spacetime Maxwell equations in terms of the electric and magnetic fields (and a nonzero conductivity during the radiation and baryon epochs). The conductivity is taken to vanish during inflation and to instantaneously assume the radiation epoch value at the inflation-radiation transition (it is also assumed to not jump at the radiation-baryon transition). The joining conditions are again determined by requiring that the equations of motion remain singularity-free at the transition hypersurface. If the conductivity after the inflation-radiation transition is large we find that the electric field after the transition is much smaller than the electric field before the transition, we also find that the magnetic field must not jump at this transition.

1.4 Some Results

The inflation scenario is, at present, not predictive, i.e., generic inflation models have more free parameters than the number of reliable large-scale observational constraints [41]. One may however use the observational data to constrain the range of free parameters [41]. In the absence of the electromagnetic field the simplest version of the exponential potential inflation model has two free parameters: the baryon epoch large-scale energy density perturbation power spectrum index n [the power spectrum $\langle\delta(k)\delta^*(k)\rangle \propto k^n$ where δ is the fractional energy density perturbation, k is spatial momentum, and n is a function of q]; and the redshift of the inflation-radiation transition $z_{R\Phi}$. In Ref. [41] we have shown that large-scale data constrains inflation models to a line in $n - z_{R\Phi}$ space.

The Maxwell-scalar field term adds a free parameter (α) and a late time field (the Maxwell field) to the model. We find that the baryon epoch proper magnetic field two-point correlation function is $\propto k^{\hat{n}}\, a^{-4}$ where a is the FLRW scale factor and $-3 < \hat{n} < 2$ ($\hat{n}$ is a function of n and α and the lower limit is a consequence of requiring that the Fourier transform to position space be infrared finite). Fourier transforming (with Gaussian smoothing) we find on a present proper scale of one-thousandth the Hubble scale that the RMS field ranges from $\lesssim 10^{-65}$G to $\gtrsim 10^{-10}$G (maybe even as large as $\sim 10^{-9}$G), with the larger values corresponding to $\hat{n} \sim -3$. [The value $\hat{n} \simeq -3$ corresponds to the almost de Sitter inflation model, with a scale-invariant ($n \simeq 1$) [74] spectrum of energy density perturbations, and a moderately large value of α (~ 20). In this case the inflation epoch effective Maxwell coupling 'constant' is large at early times. While we do not use an expansion in this coupling constant to derive our results, the inclusion of fermions during inflation might lead to a very different effective Lagrangian in the strongly coupled case. In the weakly coupled case we find, if the Maxwell field energy density is to remain below that in the homogeneous scalar field until the end of inflation, that the maximum field is $\sim 10^{-23}$G, for $\alpha \sim -2.6$ and $n = -1/2$ (i.e., energy density perturbations with more large scale power than in the scale-invariant case).] Requiring that the model reproduce the large-scale velocity data [41] and that the magnetic field have a given magnitude would constrain the allowed models to a line in the three dimensional (n, $\hat{n}$, $z_{R\Phi}$) free parameter space.

We note that our model does not take account of the electroweak phase transition; in Sec. 6.2 below we suggest that accounting for this transition is unlikely to lead to a maximum present field (on a scale of one-thousandth the Hubble radius) greater than $\sim 10^{-9}$G in this model. We emphasize that a more realistic model (that incorporates some of the physics of the standard low-energy particle physics model) must be studied and that the numbers tabulated in the previous paragraph should be considered to be preliminary estimates.

1.5 Overview

In Sec. 2 we focus on the inflation epoch of the model. The Einstein-Maxwell-scalar field action which describes the class of models is given in Sec. 2.1. In this section we also derive the linearized (synchronous and temporal gauge) equations for perturbations about a spatially homogeneous scalar field and vanishing background gauge potential in a spatially flat FLRW cosmological model. The exponential potential inflation model and the form of the Maxwell-scalar field coupling we use are described in Sec. 2.2, where we also tabulate the spatially homogeneous background field solutions and the general solution for the vector potential. In Sec. 2.3 we extend the quantum mechanical initial conditions of Refs. [70,57,58], use them to determine the constants of integration in the general vector potential solution, and record the large-scale and small-scale (relative to the Hubble scale) expansions of the vector potential. In Appendix A we define the electric and magnetic fields in terms of the vector potential (in curved spacetime), reproducing the results of Refs. [75-77], and in Appendix B we record the FLRW model version of these relations. In Sec. 2.4 we derive the inflation epoch electric and magnetic field two-point correlation functions. In this section we also evaluate the energy density carried by these perturbations and record the large-scale and small-scale forms of all three quantities.

The inflation-radiation transition is complicated by the scalar field coupling to the vector potential. In particular, we now also need to account for the transition from an effectively time-dependent Maxwell coupling 'constant' during inflation to a time-independent coupling constant now, and since we are now also interested in vector potential perturbations we need to account for the change in conductivity at the inflation-radiation transition (the conductivity changes from an essentially negligible value during inflation to a fairly large value during radiation dominance). To separate these two issues, we first focus, in Sec. 3, on a transition to late time universe with zero conductivity.

In Sec. 3.1 we determine the synchronous and temporal gauge linear perturbation equations for an ideal fluid dominated FLRW cosmological model with a Maxwell field. In Sec. 3.2 we list the homogeneous background solutions during radiation and baryon dominance and solve for the vector potential (and tabulate the small-scale and large-scale expansions of these solutions). These general solutions depend on constants of integration that are determined from joining conditions derived in a phenomenological model of the transitions. The instantaneous transition model we use is a generalization of that used in Refs. [71,41,42]. We present, in Sec. 3.3, a phenomenological prescription to account for the physics that is responsible for the scalar field developing an approximately time-independent vacuum expectation value just after the inflation-radiation transition. The joining conditions at the inflation-radiation and radiation-baryon transitions are derived, in Sec. 3.3, by requiring that the equations of motion remain singularity-free at the transitions. The joining conditions are that the electric and magnetic fields not jump at the transitions. In Sec. 3.4 we use these joining conditions to determine the constants of integration in the radiation and baryon epoch solutions for the vector potential, and tabulate expressions for the constants in the large-scale limit. In Sec. 3.5 these results are used to derive the electric and magnetic field two-point correlation functions and the energy density carried by these perturbations.

In Sec. 4 we account for the effects of a nonzero conductivity during radiation dominance. In Sec. 4.1 we solve the electric and magnetic field equations in the nonzero-conductivity radiation dominated model. The zero- and perfect-conductivity and large- and small-scale limits of these solutions are tabulated in Sec. 4.2. In Sec 4.3 we extend the inflation-radiation transition model of Sec. 3.3 to account for the jump in conductivity at this transition and derive joining conditions at the transition. At the transition the electric field jumps while the magnetic field does not. In Sec. 4.4 these joining conditions are used to determine the constants in the electric and magnetic field solutions. The large-conductivity limit of the relevant two-point functions (and the Maxwell energy density) are tabulated in Sec. 4.5 and it is shown that the electric field vanishes in the perfect-conductivity limit while the magnetic field tends to a nonzero value in this limit. In Sec. 5 the baryon epoch equations for the electric and magnetic fields are solved in the large-conductivity limit (since we have not succeeded in solving these equations in the arbitrary conductivity case). The constants of integration in these solutions are determined from joining conditions. In this section we also tabulate the baryon epoch magnetic field two-point correlation function.

To estimate the magnitude of the present large-scale magnetic field we determine, in Sec. 6.1, the position space RMS (Gaussian smoothed) field. In Sec. 6.2 we tabulate the value of the present field on a scale of one-thousandth the Hubble scale for a number of points in the restricted free parameter space (i.e., with n and $z_{R\Phi}$ related by comparison to the large-scale velocity data and with α arbitrary). In this section we also suggest that a proper accounting for the electroweak phase transition is unlikely to change our estimates of the present field strength by more than a factor of 2. In Sec. 6.3 we estimate, during inflation, the magnitude of the vector potential perturbation and the magnitude of the Maxwell energy density and show that consistency with our assumptions further constrains the allowed parameter space.

2. Inflation

In this section we study the evolution of vector potential spatial irregularities in the exponential potential scalar field dominated inflation epoch of the model.

2.1 Equations of Motion

The Einstein-Maxwell-scalar field action we consider is

$$S = \frac{m_p^2}{16\pi} \int dt\, d^3x \sqrt{-g} \left[-R + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} V(\Phi) - \frac{1}{4\bar{e}^2} f(\Phi) g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} \right] \tag{2.1}$$

where $m_p = G^{-1/2}$ is the Planck mass, G is the Newtonian gravitational constant and $\bar{e}$ is the Maxwell coupling constant; the metric ($g_{\mu\nu}$) signature is taken to be $(+ - --)$, Φ is the field responsible for inflation (the scalar field potential V and the function f are described below) and in terms of the vector potential A_μ the Maxwell field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.2}$$

The equations of motion that follow from the action (2.1) are

$$R_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2\bar{e}^2} f(\Phi) g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} \left[V(\Phi) - \frac{1}{2\bar{e}^2} f(\Phi) g^{\lambda\rho} g^{\sigma\tau} F_{\lambda\sigma} F_{\rho\tau} \right], \tag{2.3}$$

$$\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\mu \Phi \right) + \frac{1}{2} V'(\Phi) = -\frac{1}{4\bar{e}^2} f'(\Phi) g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho}, \tag{2.4}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(f(\Phi) \sqrt{-g}\, g^{\mu\lambda} g^{\rho\nu} F_{\lambda\nu} \right) = 0, \tag{2.5}$$

where a prime denotes a derivative with respect to Φ. Given eqn. (2.2), the Bianchi identity for the Maxwell field-strength tensor is identically satisfied. The stress-energy tensor is

$$\begin{aligned} \frac{8\pi}{m_p^2} T_{\mu\nu} = & \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2\bar{e}^2} f(\Phi) g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} \\ & - \frac{1}{4} g_{\mu\nu} \left[g^{\sigma\tau} \partial_\sigma \Phi \partial_\tau \Phi - V(\Phi) - \frac{1}{2\bar{e}^2} f(\Phi) g^{\lambda\rho} g^{\sigma\tau} F_{\lambda\sigma} F_{\rho\tau} \right]. \end{aligned} \tag{2.6}$$

To derive the equations of motion for the spatially homogeneous background fields and for the spatial irregularities, we linearize eqns. (2.3) – (2.5) about a spatially flat FLRW model, a spatially homogeneous scalar field and a vanishing spatially homogeneous vector potential. We work in synchronous gauge (for the metric) so the line element is taken to be

$$ds^2 = dt^2 - a^2(t)\left[\delta_{ij} - h_{ij}(t,\vec{x})\right] dx^i dx^j\,, \tag{2.7}$$

where a is the scale factor and the metric perturbations are denoted by h_{ij};

$$\Phi(t,\vec{x}) = \Phi_b(t) + \phi(t,\vec{x}) \tag{2.8}$$

where Φ_b and ϕ are the spatially homogeneous and inhomogeneous parts of the scalar field; and we work in temporal gauge (for the vector potential),

$$A_0(t,\vec{x}) = 0, \tag{2.9}$$

$$A_i(t,\vec{x}) = 0 + A_i(t,\vec{x}), \tag{2.10}$$

where the spatially homogeneous part of A_i is set to zero (since it violates spatial isotropy).

The equations of motion for the spatially homogeneous part of the fields are

$$\ddot{\Phi}_b + 3\,\frac{\dot{a}}{a}\,\dot{\Phi}_b + \frac{1}{2}V'(\Phi_b) = 0, \tag{2.11}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{12}\left[(\dot{\Phi}_b)^2 + V(\Phi_b)\right], \tag{2.12}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{12}\left[2(\dot{\Phi}_b)^2 - V(\Phi_b)\right], \tag{2.13}$$

where a dot denotes a derivative with respect to time. The vector potential does not contribute to these equations; as a result they are identical to those of the Einstein-scalar field model (Sec. VII of Ref. [65] and Sec. II of Ref. [71]).

The first order perturbation equations are

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\partial_i\partial_i\phi + \frac{1}{2}V''(\Phi_b)\phi = \frac{1}{2}\dot{h}\dot{\Phi}_b, \tag{2.14}$$

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} = 2\dot{\Phi}_b\dot{\phi} - \frac{1}{2}V'(\Phi_b)\phi, \tag{2.15}$$

$$\dot{h}_{,i} - \dot{h}_{ij,j} = \dot{\Phi}_b\partial_i\phi, \tag{2.16}$$

$$\frac{1}{a^2}(h_{ij,kk} + h_{,ij} - h_{ik,jk} - h_{jk,ik}) - 3\frac{\dot{a}}{a}\dot{h}_{ij} - \frac{\dot{a}}{a}\dot{h}\,\delta_{ij} - \ddot{h}_{ij} = \frac{1}{2}\delta_{ij}V'(\Phi_b)\phi, \tag{2.17}$$

$$\ddot{A}_i + \left(\frac{\dot{a}}{a} + \dot{\Phi}_b\frac{f'(\Phi_b)}{f(\Phi_b)}\right)\dot{A}_i - \frac{1}{a^2}\left(\partial_k\partial_k A_i - \partial_i\partial_k A_k\right) = 0, \tag{2.18}$$

$$\partial_i \dot{A}_i = 0, \tag{2.19}$$

where h is the trace of the metric perturbation and spatial indices are raised and lowered with δ_{ij}. Eqn. (2.14) governs the evolution of the scalar field perturbation, eqn. (2.15) that of the trace of the metric perturbation and eqns. (2.16) and (2.17) that of the remaining parts of the metric perturbation. The vector potential does not contribute to these equations; as a result they coincide with those of the Einstein-scalar field model [65,71]. Eqns. (2.18) and (2.19) are the vector potential perturbation equations of motion. (We note that the metric and scalar field perturbations do not contribute to these equations.) Comparing them to the vector potential equations in a model with an electrically conducting fluid (with overall charge neutrality), eqns. ($B34$) and ($B35$), we see that $\dot{\Phi}_b f'(\Phi_b)/f(\Phi_b)$ plays the role of the conductivity σ. We note that, given eqns. (2.9) and (2.10), the vector potential does not contribute to the zeroth and first order parts of the stress-energy tensor, eqn. (2.6), which is, therefore, to linear order, identical to that of the Einstein-scalar field model [65,71].

2.2 Inflation Epoch Model and Solutions

The inflation model is characterized by the scalar field potential, [65,41,58],

$$V(\Phi) = \left(\frac{6-q}{3}\right)\frac{16\pi}{m_p^2}\,\rho_{b\Phi}^{(0)}\exp\left[-\sqrt{\frac{q}{2}}\left(\Phi - \Phi^{(0)}\right)\right], \tag{2.20}$$

where $\Phi^{(0)}$ is the scalar field when the scale factor $a = a_0$, $\rho_{b\Phi}^{(0)}$ is the scalar field background energy density at the same time and q is a parameter, the relevant range being $0 < q < 2$; the scalar field potential vanishes after the transition from inflation to radiation dominance. For the function f in eqn. (2.1) we take

$$f(\Phi) = e^{\alpha\Phi}, \tag{2.21}$$

where α is a parameter. This form is chosen for simplicity; it would be of interest to study models with different functional forms.

The vector potential does not appear in the equations of motion for the spatially homogeneous fields, eqns. (2.11) – (2.13). The solutions of these equations for the exponential potential inflation model, eqn. (2.20), are, [65,41],

$$a(t) = a_0 \left[1 + M(t - t_0)\right]^{2/q} , \tag{2.22}$$

$$\Phi_b(t) = \Phi^{(0)} + \sqrt{\frac{8}{q}} \ln \left[1 + M(t - t_0)\right] , \tag{2.23}$$

where $t = t_0$ when $a = a_0$ and we have defined

$$M = \frac{q}{2} \left(\frac{8\pi}{3m_p^2} \rho_{b\Phi}^{(0)} \right)^{1/2} . \tag{2.24}$$

We note that eqns. (2.22) and (2.23) describe a time-dependent stable fixed point (for $q < 2$) [65], and that the spatially homogeneous energy density of the scalar field is given by

$$\rho_{b\Phi} = \rho_{b\Phi}^{(0)} \left(\frac{a_0}{a} \right)^q , \tag{2.25}$$

while the corresponding equation of state relates the scalar field background pressure to the scalar field background energy density through

$$p_{b\Phi} = - \left(\frac{3 - q}{3} \right) \rho_{b\Phi} . \tag{2.26}$$

The vector potential perturbation does not influence the evolution of scalar field and metric linear perturbations, eqns. (2.14) – (2.17). The solutions of these equations, for the exponential potential inflation model, are given in Ref. [41].

To solve eqns. (2.18) and (2.19) for the vector potential perturbation we use the techniques developed in Refs. [69,41]. We Fourier expand the vector potential inhomogeneity,

$$A_i(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[A_i(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} + A_i^*(t, \vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right] , \tag{2.27}$$

where $\vec{k}$ is spatial coordinate momentum ($-\infty \leq k_i \leq \infty$), $\vec{k} \cdot \vec{x} = \delta_{ij} k_i x_j$ and we shall use

k to denote $|k_i k_i|^{1/2}$. We also change from the independent variable t (time) to x,

$$e^x = t - t_0 + M^{-1}. \tag{2.28}$$

Considering a plane wave with spatial momentum vector $\vec{k}$ and choosing the x^1 axis to lie along this spatial momentum direction (and denoting the transverse directions by $x^I, I = 2, 3$) we find, on using eqns. (2.21) – (2.23), (2.27) and (2.28), that the spatial momentum space version of eqns. (2.18) and (2.19) are

$$A_1''(x) + \left(\frac{2-q}{q} + \alpha\sqrt{\frac{8}{q}}\right) A_1'(x) = 0, \tag{2.29}$$

$$A_I''(x) + \left(\frac{2-q}{q} + \alpha\sqrt{\frac{8}{q}}\right) A_I'(x) + \left(\frac{2-q}{q}\right)^2 \lambda^2 e^{-2(2-q)x/q} A_I(x) = 0, \tag{2.30}$$

$$A_1'(x) = 0, \tag{2.31}$$

where we have defined [41],

$$\lambda = \left(\frac{q}{2-q}\right) \frac{k}{a_0 M^{2/q}}. \tag{2.32}$$

The solution of eqns. (2.29) and (2.31) is

$$A_1(x) = C_1, \tag{2.33}$$

where C_1 is a k dependent constant of integration, and the solution of eqn. (2.30) is

$$A_I(x) = e^{-(2-q)\hat{\nu}x/q} \left[C_{I+} H_{\hat{\nu}}^{(1)}\left(\lambda\, e^{-(2-q)x/q}\right) + C_{I-} H_{\hat{\nu}}^{(2)}\left(\lambda\, e^{-(2-q)x/q}\right)\right], \tag{2.34}$$

where $C_{I\pm}$ are k dependent constants of integration, $H_{\hat{\nu}}^{(i)}$ are Hankel functions and

$$\hat{\nu} = \frac{1}{2}\left(1 + \alpha\,\frac{q}{2-q}\sqrt{\frac{8}{q}}\right). \tag{2.35}$$

The choice of synchronous gauge still allows four time-independent general coordinate transformations. Their effect on the metric and scalar field perturbations has been examined in Ref. [41]. It is straightforward to establish that to linear order the vector potential perturbation A_i is invariant to the remnants of general coordinate invariance in synchronous gauge; as a result the C_1 solution in eqn. (2.33) and the $C_{I\pm}$ solutions in eqn. (2.34) are general coordinate invariant. The choice of temporal gauge does not completely fix the $U(1)$ Maxwell gauge symmetry — there is one remaining time-independent transformation,

$$\delta_M A_1(x,k) = ik\Lambda(k), \tag{2.36}$$

$$\delta_M A_I(x,k) = 0, \tag{2.37}$$

where the $U(1)$ gauge parameter Λ is time (x) independent. From eqn. (2.36) we find that

$$\Lambda(k) = -\frac{i}{k}C_1 \tag{2.38}$$

generates the solution of eqn. (2.33), while from eqn. (2.37) we see that the solutions of eqn. (2.34) are $U(1)$ gauge-invariant.

2.3 Asymptotics and Initial Conditions

It is useful to note that

$$\lambda\, e^{-(2-q)x/q} = \left(\frac{2}{2-q}\right)\frac{k}{aH} = -k\int^t \frac{dt'}{a(t')} = -k\tau, \tag{2.39}$$

where $H = (\dot{a}/a)$ is the Hubble parameter and τ is conformal time.

Focussing on a plane wave with propagation vector along the x^1 axis and expanding eqn. (2.1) to quadratic order in the perturbations, we find that the quadratic order action for the transverse vector potential perturbation is

$$S_{A_I} = \frac{m_p^2}{16\pi}\int a\,dt\,d^3x\,\frac{f(\Phi_b)}{\bar{e}^2}\,\delta_{IJ}\left[\frac{1}{2}\dot{A}_I\dot{A}_J - \frac{1}{2a^2}\partial_1 A_I\partial_1 A_J\right]. \tag{2.40}$$

Defining a rescaled vector potential by

$$\mathcal{A}_I = \sqrt{f(\Phi_b)}\,A_I, \tag{2.41}$$

we find eqn. (2.40) becomes

$$S_{A_I} = \frac{m_p^2}{16\pi}\int a\,dt\,d^3x\,\frac{\delta_{IJ}}{\bar{e}^2}\left[\frac{1}{2}\dot{\mathcal{A}}_I\dot{\mathcal{A}}_J - \frac{1}{2a^2}\partial_1\mathcal{A}_I\partial_1\mathcal{A}_J\right.$$

$$+\frac{\alpha}{4}(2-q)\sqrt{\frac{q}{2}}\left\{1+\alpha\frac{q}{2-q}\sqrt{\frac{2}{q}}\right\}H^2\mathcal{A}_I\mathcal{A}_J\Bigg] \tag{2.42}$$

where we have discarded a surface term and used the background solutions of Sec. 2.2.

On small enough spatial scales, i.e., for spatial momenta that satisfy

$$\left(\frac{k}{aH}\right)^2 \gg \left|\alpha\left(\frac{2-q}{2}\right)\sqrt{\frac{q}{2}}\left[1+\alpha\frac{q}{2-q}\sqrt{\frac{2}{q}}\right]\right|, \tag{2.43}$$

the 'mass' term in eqn. (2.42) may be neglected (compared to the spatial gradient term) and in terms of conformal time, eqn. (2.39), this part of the action reduces to

$$S_{A_I} = \frac{m_p^2}{16\pi}\int d\tau\, d^3x\, \frac{\delta_{IJ}}{\bar{e}^2}\left[\frac{1}{2}\frac{d\mathcal{A}_I}{d\tau}\frac{d\mathcal{A}_J}{d\tau} - \frac{1}{2}\partial_1\mathcal{A}_I\partial_1\mathcal{A}_J\right], \tag{2.44}$$

which is of the standard form, [70,57,41,58]; as a result the appropriate quantum mechanical initial condition is

$$\lim_{k\gg aH}\mathcal{A}_I(k) = \left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{\bar{e}}{\sqrt{2k}}\; e^{-ik\tau}, \tag{2.45}$$

or, from eqn. (2.41),

$$\lim_{k\gg aH}A_I(k) = \left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{\bar{e}}{\sqrt{2kf(\Phi_b)}}\; e^{-ik\tau}. \tag{2.46}$$

If we replace the constants of integration in eqn. (2.34), $C_{I\pm}$, by the x independent constants of integration $\widetilde{C}_{I\pm}$ defined by

$$C_{I\pm} = \left(\frac{16\pi}{m_p^2}\right)^{1/2}\bar{e}\,\frac{\sqrt{\pi}}{2}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{(q/2)}}\left(\frac{k}{\lambda}\right)^{-(1+\alpha\sqrt{2q})/2} e^{-\alpha\Phi^{(0)}/2}$$
$$\times\,(a_0)^{\alpha\sqrt{(q/2)}}\, e^{\pm i(\hat{\nu}+1/2)\pi/2}\,\widetilde{C}_{I\pm}, \tag{2.47}$$

we find in the short wavelength limit (when $k\to\infty$) that eqn. (2.34) reduces to

$$A_I(k,t) = \left(\frac{16\pi}{m_p^2}\right)^{1/2}\frac{\bar{e}}{\sqrt{2kf(\Phi_b)}}\left[\widetilde{C}_{I+}\,e^{-ik\tau}+\widetilde{C}_{I-}\,e^{ik\tau}\right]; \tag{2.48}$$

for this to agree with eqn. (2.46) we must require

$$\widetilde{C}_{I+} = 1 \qquad \text{and} \qquad \widetilde{C}_{I-} = 0. \tag{2.49}$$

With this initial condition the leading terms in the large-scale asymptotic expansion of the vector potential perturbation, when $\hat{\nu} > 0$, are

$$A_I(k,x) = C_1^+ \, k^{-\hat{\nu}} + C_2^+ \, k^{2-\hat{\nu}} \, e^{-2(2-q)x/q} + C_3^+ \, k^{\hat{\nu}} \, e^{-2(2-q)\hat{\nu}x/q} + \cdots , \tag{2.50}$$

where the ellipses denote terms that are down by positive powers of $k/(aH)$ and

$$\begin{aligned} C_1^+ =& \left(\frac{16\pi}{m_p^2}\right)^{1/2} \bar{e}\,\sqrt{\pi}\,2^{\hat{\nu}-1}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{(q/2)}}\left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{(q/2)}/(2-q)} \\ &\times \frac{\csc(\hat{\nu}\pi)}{\Gamma(1-\hat{\nu})}\, e^{-\alpha\Phi^{(0)}/2}\,(a_0)^{\alpha\sqrt{(q/2)}} e^{i(\hat{\nu}-1/2)\pi/2}, \end{aligned} \tag{2.51}$$

$$\begin{aligned} C_2^+ =& \left(\frac{16\pi}{m_p^2}\right)^{1/2} \bar{e}\,\sqrt{\pi}\,2^{\hat{\nu}-3}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{(q/2)}}\left(\frac{k}{\lambda}\right)^{[-2+\alpha q\sqrt{(q/2)}/(2-q)]} \\ &\times \frac{\csc(\hat{\nu}\pi)}{\Gamma(2-\hat{\nu})}\, e^{-\alpha\Phi^{(0)}/2}\,(a_0)^{\alpha\sqrt{(q/2)}} e^{i(\hat{\nu}-5/2)\pi/2}, \end{aligned} \tag{2.52}$$

$$\begin{aligned} C_3^+ =& \left(\frac{16\pi}{m_p^2}\right)^{1/2} \bar{e}\,\sqrt{\pi}\,2^{-(\hat{\nu}+1)}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{(q/2)}}\left(\frac{k}{\lambda}\right)^{-[1+\alpha(4-q)\sqrt{(q/2)}/(2-q)]} \\ &\times \frac{\csc(\hat{\nu}\pi)}{\Gamma(1+\hat{\nu})}\, e^{-\alpha\Phi^{(0)}/2}\,(a_0)^{\alpha\sqrt{(q/2)}} e^{-i(\hat{\nu}-3/2)\pi/2}, \end{aligned} \tag{2.53}$$

where Γ is the gamma function. We note that the leading large-scale term in eqn. (2.50) is time-independent and that the spatial momentum dependence of this term is α dependent. When $\hat{\nu} < 0$ the large-scale asymptotic expansion of eqn. (2.50) is replaced by

$$A_I(k,x) = C_1^- \, k^{\hat{\nu}} \, e^{-2(2-q)\hat{\nu}x/q} + C_2^- \, k^{\hat{\nu}+2} \, e^{-2(2-q)(\hat{\nu}+1)x/q} + C_3^- \, k^{-\hat{\nu}} + \cdots , \tag{2.54}$$

where we have defined

$$C_1^- = C_3^+, \tag{2.55}$$

$$\begin{aligned} C_2^- =& \left(\frac{16\pi}{m_p^2}\right)^{1/2} \bar{e}\,\sqrt{\pi}\,2^{-(\hat{\nu}+3)}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{(q/2)}}\left(\frac{k}{\lambda}\right)^{-[3+\alpha(4-q)\sqrt{(q/2)}/(2-q)]} \\ &\times \frac{\csc(\hat{\nu}\pi)}{\Gamma(2+\hat{\nu})}\, e^{-\alpha\Phi^{(0)}/2}\,(a_0)^{\alpha\sqrt{(q/2)}} e^{-i(\hat{\nu}+1/2)\pi/2}, \end{aligned} \tag{2.56}$$

$$C_3^- = C_1^+, \tag{2.57}$$

and C_1^+ and C_3^+ are given in eqns. (2.51) and (2.53). It is interesting to note that in this case the leading large-scale term in eqn. (2.54) is time-dependent and that the time dependence as well as the spatial momentum dependence of this term are α dependent.

2.4 Derived Fields

We find that the correlation functions of the nonvanishing components of the Maxwell field-strength tensor are given by

$$\begin{aligned}\langle F_{0I}(k,x)F^*_{0I}(k,x)\rangle = &\left(\frac{2-q}{q}\right)^2 |C_{I+}|^2\lambda^2\, e^{-2[1+(2-q)(\hat\nu+1)/q]x}\\ &\times H^{(1)}_{\hat\nu-1}\left(\lambda\, e^{-(2-q)x/q}\right) H^{(2)}_{\hat\nu-1}\left(\lambda\, e^{-(2-q)x/q}\right), \qquad (2.58)\\ \langle F_{1I}(k,x)F^*_{1I}(k,x)\rangle =&|C_{I+}|^2 k^2\, e^{-2(2-q)\hat\nu x/q} H^{(1)}_{\hat\nu}\left(\lambda\, e^{-(2-q)x/q}\right) H^{(2)}_{\hat\nu}\left(\lambda\, e^{-(2-q)x/q}\right) \qquad (2.59)\end{aligned}$$

[there is no implicit summation over the index I on the left hand sides of eqns. (2.58) and (2.59) and in similar equations below]. We note from eqns. $(B19)$ and $(B20)$ [and the discussion below them and below eqns. $(B37)$ and $(B38)$] that the proper transverse electric field two-point function $a^{-2}\left\langle E_I(k,a)E^*_I(k,a)\right\rangle = a^{-2}\left\langle F_{0I}(k,a)F^*_{0I}(k,a)\right\rangle$, and a proper transverse magnetic field two-point function $a^{-2}\langle B_2(k,a)B^*_2(k,a)\rangle = a^{-4}\langle F_{13}(k,a)F^*_{13}(k,a)\rangle$.

Expanding these expressions we find that the leading small-scale terms are

$$\begin{aligned}\langle F_{0I}(k,a)F^*_{0I}(k,a)\rangle =&\frac{8\pi}{m_p^2}\,\bar e^2\, e^{-\alpha\Phi^{(0)}}(a_0)^{\alpha\sqrt{2q}} k\, a^{-(2+\alpha\sqrt{2q})}, \qquad (2.60)\\ \langle F_{1I}(k,a)F^*_{1I}(k,a)\rangle =&\frac{8\pi}{m_p^2}\,\bar e^2\, e^{-\alpha\Phi^{(0)}}(a_0)^{\alpha\sqrt{2q}} k\, a^{-\alpha\sqrt{2q}}. \qquad (2.61)\end{aligned}$$

On small scales the proper transverse electric and magnetic fields scale as $a^{-(4+\alpha\sqrt{2q})/2}$.

When $\hat\nu > 0$ the leading large-scale terms in the correlation functions are

$$\begin{aligned}&\langle F_{0I}(k,a)F^*_{0I}(k,a)\rangle =\\ &\frac{16\pi}{m_p^2}\,\bar e^2\, e^{-\alpha\Phi^{(0)}}(a_0)^{\alpha\sqrt{2q}}\pi\, 2^{2(\hat\nu-2)}\left(\frac{q}{2-q}\right)^{(2-q-\alpha\sqrt{2q})}\left(\frac{k}{\lambda}\right)^{[\alpha q\sqrt{2q}/(2-q)-q]}\\ &\times\frac{\csc^2(\hat\nu\pi)}{\Gamma^2(2-\hat\nu)}\, k^{2(2-\hat\nu)} a^{(q-4)} \qquad (2.62)\\ &\times\left[1+2^{(3-2\hat\nu)}\left(\frac{2-q}{q}\right)^{(1-\hat\nu)(2-q)}\cos(\hat\nu\pi)\,\frac{\Gamma(2-\hat\nu)}{\Gamma(\hat\nu)}\left(\frac{k}{\lambda}\right)^{q(1-\hat\nu)} k^{2(\hat\nu-1)}\, a^{(1-\hat\nu)(2-q)}\right.\\ &\left.\quad+\,2^{4(1-\hat\nu)}\left(\frac{2-q}{q}\right)^{2(1-\hat\nu)(2-q)}\frac{\Gamma^2(2-\hat\nu)}{\Gamma^2(\hat\nu)}\left(\frac{k}{\lambda}\right)^{2q(1-\hat\nu)} k^{4(\hat\nu-1)}\, a^{2(1-\hat\nu)(2-q)}+\cdots\right],\end{aligned}$$

$$\langle F_{1I}(k,a)F^*_{1I}(k,a)\rangle = \frac{16\pi}{m_p^2}\,\bar{e}^2\,e^{-\alpha\Phi^{(0)}}(a_0)^{\alpha\sqrt{2q}}\pi\,2^{2(\hat{\nu}-1)}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}}$$
$$\times\left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)}\frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1-\hat{\nu})}\,k^{2(1-\hat{\nu})}+\cdots. \tag{2.63}$$

Similarly, when $\hat{\nu} < 0$ we have

$$\langle F_{0I}(k,a)F^*_{0I}(k,a)\rangle = \frac{16\pi}{m_p^2}\,\bar{e}^2\,e^{-\alpha\Phi^{(0)}}(a_0)^{\alpha\sqrt{2q}}\pi\,2^{-2\hat{\nu}}\left(\frac{q}{2-q}\right)^{\alpha\sqrt{2q}}$$
$$\times\left(\frac{k}{\lambda}\right)^{-\alpha q\sqrt{2q}/(2-q)}\frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(\hat{\nu})}\,k^{2\hat{\nu}}\,a^{-2(1+\alpha\sqrt{2q})}+\cdots, \tag{2.64}$$
$$\langle F_{1I}(k,a)F^*_{1I}(k,a)\rangle = \frac{16\pi}{m_p^2}\,\bar{e}^2\,e^{-\alpha\Phi^{(0)}}(a_0)^{\alpha\sqrt{2q}}\pi\,2^{-2(\hat{\nu}+1)}\left(\frac{q}{2-q}\right)^{(2-q+\alpha\sqrt{2q})}$$
$$\times\left(\frac{k}{\lambda}\right)^{-[q+\alpha q\sqrt{2q}/(2-q)]}\frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1+\hat{\nu})}\,k^{2(1+\hat{\nu})}a^{-(2-q+2\alpha\sqrt{2q})}+\cdots. \tag{2.65}$$

From these equations we see that the large-scale proper transverse electric field scales like $a^{-(3-q/2)}$ [for $1 \le \hat{\nu}$] and $a^{-(2+\alpha\sqrt{2q})}$ [for $\hat{\nu} \le 1$], while the large-scale proper transverse magnetic field scales as a^{-2} [for $0 \le \hat{\nu}$] and $a^{-(6-q+2\alpha\sqrt{2q})/2}$ [for $\hat{\nu} \le 0$].

Defining the energy density spectrum, $\epsilon_I(k,t)$, of the transverse vector potential,

$$\epsilon_I(k,t) = \frac{m_p^2}{8\pi}\,\frac{f(\Phi_b)}{\bar{e}^2a^2}\left[\langle F_{0I}(k,t)F^*_{0I}(k,t)\rangle + \frac{1}{a^2}\langle F_{1I}(k,t)F^*_{1I}(k,t)\rangle\right] \tag{2.66}$$

(there is no implicit summation over the index I on the right hand side of this equation), one may show that the expectation value of the quadratic order hamiltonian is

$$\langle H_2(t)\rangle = \int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\,a^3(t)\,[\epsilon_2(k,t)+\epsilon_3(k,t)] = \int_{-\infty}^{\infty} d^3x\,a^3(t)\left\langle T_{00}^{(M,T)}(t,\vec{x})\right\rangle, \tag{2.67}$$

where $T_{00}^{(M,T)}$ is the time-time component of the transverse Maxwell part of the stress-energy tensor, eqn. (2.6); $[\epsilon_I(k)]^2$ plays a role somewhat similar to that played by $\rho_b^2\,\langle\delta(k)\delta^*(k)\rangle$ in the case of fluid perturbations (here δ is the fluid fractional energy density perturbation). The first term in the square parentheses in eqn. (2.66) is the electric contribution and the second term is the magnetic contribution. Using the results of Appendix B, it may be shown that the energy density expression of eqn. (2.66) implies that the interacting scalar-vector system is equivalent to an electromagnetic field in a medium with a dielectric 'constant' ϵ and a magnetic permeability μ that obey $\epsilon = [m_p^2/(4\bar{e}^2)]f(\Phi_b) = \mu^{-1}$.

On small scales we find

$$\epsilon_I(k,t) = 2\,\frac{k}{a^4}, \tag{2.68}$$

where, from eqns. (2.60) and (2.61), we note that both the electric and magnetic terms of eqn. (2.66) contribute.

On large scales when $\hat{\nu} > 0$ we have

$$\epsilon_I(k,t) = 2A_1 \frac{k^{2(1-\hat{\nu})}}{a^{(4-\alpha\sqrt{2q})}} + 2A_2 \frac{k^{2\hat{\nu}}}{a^{(4+\alpha\sqrt{2q})}} + \cdots, \tag{2.69}$$

where we have defined

$$A_1 = \pi\, 2^{2(\hat{\nu}-1)} \left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)} \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1-\hat{\nu})}, \tag{2.70}$$

$$A_2 = \pi\, 2^{-2\hat{\nu}} \left(\frac{q}{2-q}\right)^{\alpha\sqrt{2q}} \left(\frac{k}{\lambda}\right)^{-\alpha q\sqrt{2q}/(2-q)} \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(\hat{\nu})}; \tag{2.71}$$

the first term in eqn. (2.69) is the magnetic field contribution of eqn. (2.63) while the second term is the electric field contribution of eqn. (2.62). We note that when $\hat{\nu} < 1/2$ ($\alpha < 0$) the second term in eqn. (2.69) dominates, while the first term dominates for $\hat{\nu} > 1/2$ ($\alpha > 0$). In both cases the Maxwell part of the large-scale energy density grows slower than a^{-4} as a is decreased. It is pleasing to note that when $\alpha = 0$ ($\hat{\nu} = 1/2$)

$$A_1 = \frac{1}{2} = A_2 \tag{2.72}$$

and eqn. (2.69) reduces to eqn. (2.68) [as it must, since the Maxwell part of the action is now conformally invariant and no longer recognizes as special the scale set by the Hubble radius]. When $\hat{\nu} < 0$ the large-scale part of the energy density is

$$\epsilon_I(k,t) = 2A_2 \frac{k^{2\hat{\nu}}}{a^{(4+\alpha\sqrt{2q})}} + \cdots, \tag{2.73}$$

where A_2 is defined in eqn. (2.71); we note that this term is the electric field contribution of eqn. (2.64) and that the magnetic field contribution of eqn. (2.65) is suppressed. We also note that (since $\alpha < 0$ when $\hat{\nu} < 1/2$) the large-scale energy density grows slower than a^{-4} as a is decreased when $\hat{\nu} < 0$.

3. Radiation Dominance and Baryon Dominance: 'Free' Photons

In this section we study vector potential irregularities in the radiation and baryon dominated epochs. To isolate the effects of the 'freezing' of the effective Maxwell coupling constant at the reheating transition we first study a zero conductivity model. The propagation of 'free' photons during radiation dominance has also been studied in Ref. [78].

3.1 Equations of Motion

Besides the Maxwell field, the matter content of the radiation dominated and baryon dominated epochs will be described by a linear ideal fluid with stress-energy tensor

$$T^{\mu\nu}_{(F)} = (\rho + p) u^\mu u^\nu - g^{\mu\nu} p, \tag{3.1}$$

where ρ and p are the energy density and pressure of the fluid and u^μ is the fluid coordinate peculiar velocity, and with an equation of state

$$\frac{p}{\rho} = \nu = \frac{dp}{d\rho}, \tag{3.2}$$

where $\sqrt{\nu}$ is the speed of sound which is taken to be constant.

We assume that the only coupling between the fluid and the Maxwell field is through gravitation, so the equations of motion are

$$R_{\mu\nu} = \frac{8\pi}{m_p^2}\left[T_{(T)\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{(T)}\right], \tag{3.3}$$

$$T_{(F)\mu}{}^{\nu}{}_{;\nu} = 0 \tag{3.4}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\lambda} g^{\rho\nu} F_{\lambda\nu}\right) = 0, \tag{3.5}$$

where $T_{(T)}$ is the trace of the total stress-energy tensor which is given by

$$T_{(T)\mu\nu} = T_{(F)\mu\nu} + T_{(M)\mu\nu}, \tag{3.6}$$

where the Maxwell stress-energy tensor is

$$\frac{8\pi}{m_p^2} T_{(M)\mu\nu} = -\frac{1}{2e^2} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} + \frac{1}{8e^2} g_{\mu\nu} g^{\lambda\rho} g^{\sigma\tau} F_{\lambda\sigma} F_{\rho\tau}, \tag{3.7}$$

and e is the Maxwell coupling constant during radiation and baryon dominance.

To derive the equations of motion for the spatially homogeneous background fields and for the spatial irregularities we linearize eqns. (3.3) – (3.5) about a spatially flat FLRW model, a vanishing spatially homogeneous vector potential and a spatially homogeneous fluid background. We work in synchronous gauge with the metric expansion given in eqn. (2.7) and in temporal gauge with the vector potential expansion given in eqns. (2.9) and (2.10). The expansion for the fluid variables is

$$\rho(t, \vec{x}) = \rho_b(t) \left[1 + \delta(t, \vec{x})\right], \tag{3.8}$$

$$p(t, \vec{x}) = p_b(t) + \nu \rho_b(t) \delta(t, \vec{x}), \tag{3.9}$$

$$u^0(t, \vec{x}) = 1, \tag{3.10}$$

$$u^i(t, \vec{x}) = 0 + u^i(t, \vec{x}), \tag{3.11}$$

where ρ_b and p_b are the homogeneous background fluid energy density and pressure and δ is the fractional perturbation in the fluid energy density.

From eqns. (2.9) and (2.10) we see that the leading contribution to the Maxwell stress-energy tensor, eqn. (3.7), is quadratic in the perturbations and so does not contribute to the Einstein equation, eqn. (3.3), which to linear order is identical to that of the Einstein-ideal fluid model. To linear order in the perturbations the only new equations (compared to those of the Einstein-ideal fluid model, [62,69]) are those that follow from eqn. (3.5) —

$$\ddot{A}_i + \frac{\dot{a}}{a} \dot{A}_i - \frac{1}{a^2} \left(\partial_k \partial_k A_i - \partial_i \partial_k A_k\right) = 0, \tag{3.12}$$

$$\partial_i \dot{A}_i = 0; \tag{3.13}$$

these are the equations of motion for the vector potential perturbation. We shall work in spatial Fourier space, eqn. (2.27), and choose the x^1 axis to lie along the momentum vector $\vec{k}$ of the perturbation of interest; eqns. (3.12) and (3.13) then become

$$\ddot{A}_1 + \frac{\dot{a}}{a}\dot{A}_1 = 0, \tag{3.14}$$

$$\ddot{A}_I + \frac{\dot{a}}{a}\dot{A}_I + \frac{k^2}{a^2}A_I = 0, \tag{3.15}$$

$$\dot{A}_1 = 0. \tag{3.16}$$

3.2 Solutions

The solutions for the background fields in the radiation and baryon dominated epochs, in the exponential potential inflation model, with the spatially homogeneous energy density hypersurface instantaneous transition prescription (discussed below), are given in Ref. [41]. During radiation dominance the scale factor is

$$a(t) = a_1 \left[1 + M_R(t - t_{R\Phi})\right]^{1/2}, \tag{3.17}$$

where $t_{R\Phi}$ is the time of equal radiation and homogeneous scalar field energy density,

$$a_1 = a_0 \left[1 + M(t_{R\Phi} - t_0)\right]^{2/q}, \tag{3.18}$$

M is given in eqn. (2.24), and we have defined

$$M_R = 2\left(\frac{8\pi}{3m_p^2}\rho_{bR}^{(1)}\right)^{1/2}, \tag{3.19}$$

where $\rho_{bR}^{(1)}$ is the spatially homogeneous energy density (in radiation) at the inflation-radiation transition at $a = a_1$. In the baryon dominated epoch we have

$$a(t) = a_2 \left[1 + M_B(t - t_{BR})\right]^{2/3}, \tag{3.20}$$

where t_{BR} is the time of equal baryon and radiation energy density,

$$a_2 = a_0 \left[1 + M(t_{R\Phi} - t_0)\right]^{2/q} \left[1 + M_R(t_{BR} - t_{R\Phi})\right]^{1/2} , \tag{3.21}$$

and

$$M_B = \frac{3}{2} \left(\frac{8\pi}{3m_p^2} \rho_{bB}^{(2)} \right)^{1/2} , \tag{3.22}$$

where $\rho_{bB}^{(2)}$ is the spatially homogeneous energy density (in baryons) at the radiation-baryon transition when $a = a_2$.

Using the above expressions we may integrate eqns. (3.14) – (3.16) to find, during radiation dominance (related solutions are given in Ref. [78])

$$A_1^{(R)}(x) = C_1^{(R)}, \tag{3.23}$$

$$A_I^{(R)}(x) = e^{x/4} \left[C_{I+}^{(R)} H_{1/2}^{(1)} \left(\sqrt{3}\lambda_R\, e^{x/2} \right) + C_{I-}^{(R)} H_{1/2}^{(2)} \left(\sqrt{3}\lambda_R\, e^{x/2} \right) \right] , \tag{3.24}$$

where $C_1^{(R)}$ and $C_{I\pm}^{(R)}$ are k dependent constants of integration,

$$e^x = t - t_{R\Phi} + (M_R)^{-1}, \tag{3.25}$$

$$\lambda_R = \frac{2}{\sqrt{3}} \frac{k}{a_1 (M_R)^{1/2}}. \tag{3.26}$$

During baryon dominance the solutions of eqns. (3.14) – (3.16) are given by

$$A_1^{(B)}(x) = C_1^{(B)}, \tag{3.27}$$

$$A_I^{(B)}(x) = e^{x/6} \left[C_{I+}^{(B)} H_{1/2}^{(1)} \left(\widetilde{\lambda}_B\, e^{x/3} \right) + C_{I-}^{(B)} H_{1/2}^{(2)} \left(\widetilde{\lambda}_B\, e^{x/3} \right) \right] , \tag{3.28}$$

where $C_1^{(B)}$ and $C_{I\pm}^{(B)}$ are k dependent constants of integration,

$$e^x = t - t_{BR} + (M_B)^{-1}, \tag{3.29}$$

$$\widetilde{\lambda}_B = 3 \frac{k}{a_2 (M_B)^{2/3}}. \tag{3.30}$$

It is straightforward to establish that, to linear order in the perturbations, the $C_1^{(R)}$, $C_{I\pm}^{(R)}$, $C_1^{(B)}$ and $C_{I\pm}^{(B)}$ solutions in eqns. (3.23), (3.24), (3.27) and (3.28) are general coordinate invariant. It may also be shown that the $C_1^{(R)}$ and $C_1^{(B)}$ solutions are $U(1)$ gauge-dependent while the $C_{I\pm}^{(R)}$ and $C_{I\pm}^{(B)}$ solutions are $U(1)$ gauge-invariant.

In the small-scale limit the expression in eqn. (3.24) oscillates with a time-independent preexponential amplitude, while in the large-scale limit we find

$$A_I^{(R)}(k,x) = \check{C}_1^{(R)}\, k^{-1/2} + \check{C}_2^{(R)}\, k^{1/2}\, e^{x/2} + \check{C}_3^{(R)}\, k^{3/2}\, e^{x} + \cdots, \tag{3.31}$$

where the ellipses denote terms that are down by positive powers of $k/(aH)$ and

$$\check{C}_1^{(R)} = -\, i \left(\frac{2}{\sqrt{3}\pi}\right)^{1/2} \left(\frac{k}{\lambda_R}\right)^{1/2} \left(C_{I+}^{(R)} - C_{I-}^{(R)}\right), \tag{3.32}$$

$$\check{C}_2^{(R)} = \left(\frac{2\sqrt{3}}{\pi}\right)^{1/2} \left(\frac{\lambda_R}{k}\right)^{1/2} \left(C_{I+}^{(R)} + C_{I-}^{(R)}\right), \tag{3.33}$$

$$\check{C}_3^{(R)} = i \left(\frac{3\sqrt{3}}{2\pi}\right)^{1/2} \left(\frac{\lambda_R}{k}\right)^{3/2} \left(C_{I+}^{(R)} - C_{I-}^{(R)}\right). \tag{3.34}$$

On small scales the baryon epoch solution for the vector potential, eqn. (3.28), also oscillates with a time-independent preexponential amplitude, while in the large-scale limit we find

$$A_I^{(B)}(k,x) = \check{C}_1^{(B)}\, k^{-1/2} + \check{C}_2^{(B)}\, k^{1/2}\, e^{x/3} + \check{C}_3^{(B)}\, k^{3/2}\, e^{2x/3} + \cdots, \tag{3.35}$$

where we have defined

$$\check{C}_1^{(B)} = -\, i \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{k}{\widetilde{\lambda}_B}\right)^{1/2} \left(C_{I+}^{(B)} - C_{I-}^{(B)}\right), \tag{3.36}$$

$$\check{C}_2^{(B)} = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{\widetilde{\lambda}_B}{k}\right)^{1/2} \left(C_{I+}^{(B)} + C_{I-}^{(B)}\right), \tag{3.37}$$

$$\check{C}_3^{(B)} = \frac{i}{\sqrt{2\pi}} \left(\frac{\widetilde{\lambda}_B}{k}\right)^{3/2} \left(C_{I+}^{(B)} - C_{I-}^{(B)}\right). \tag{3.38}$$

It is interesting to note that the leading large-scale contribution to the vector potential is time-independent during radiation and baryon dominance, eqns. (3.31) and (3.35).

3.3 Transition Model and Derivation of Joining Conditions

The expressions derived for the vector potential depend on constants of integration. These constants will be determined from joining conditions at the transitions, derived by requiring that the equations of motion remain singularity-free on a transition hypersurface, [71,66]. (The joining conditions thus determine the amplitudes of the relevant late-time quantities.) We model a transition between two epochs as occurring instantaneously on a spatially homogeneous local (total) energy density hypersurface, i.e., we require that the equation of state and the "speed of sound" change instantaneously on this hypersurface [71,41,42].

The joining conditions for all but the vector potential perturbation have been derived in Ref. [71]. The equations of motion for this field are

$$\ddot{A}_1 + \left(\frac{\dot{a}}{a} + \dot{\Phi}_b \frac{f'(\Phi_b)}{f(\Phi_b)}\right) \dot{A}_1 = 0, \tag{3.39}$$

$$\ddot{A}_I + \left(\frac{\dot{a}}{a} + \dot{\Phi}_b \frac{f'(\Phi_b)}{f(\Phi_b)}\right) \dot{A}_I - \frac{1}{a^2}\partial_1^2 A_I = 0, \tag{3.40}$$

$$\partial_1 \dot{A}_1 = 0, \tag{3.41}$$

where $f'(\Phi_b) = 0$ during radiation and baryon dominance. Since A_i is general coordinate invariant to lowest order in the perturbations, eqns. (3.39) – (3.41) are unaffected by the transformation to the spatially homogeneous energy density transition spatial hypersurface [71]. The vector potential is a field-space coordinate and so must not jump at a transition

$$A_i^{(R)}(t_{R\Phi}, \vec{x}) = A_i^{(\Phi)}(t_{R\Phi}, \vec{x}), \tag{3.42}$$

$$A_i^{(B)}(t_{BR}, \vec{x}) = A_i^{(R)}(t_{BR}, \vec{x}), \tag{3.43}$$

where $t_{R\Phi}$ and t_{BR} are the synchronous gauge times of the inflation-radiation and radiation-baryon transitions and $A_i^{(\Phi)}$ is the vector potential during inflation. Eqns. (3.42) and (3.43) ensure that the left hand side of eqn. (3.41) is singularity-free at the transitions.

During radiation dominance and during baryon dominance the $\dot{\Phi}_b f'/f$ term in eqns. (3.39) and (3.40) vanishes; given eqn. (3.43) and given that the Hubble parameter does not jump at the transitions [71], to ensure that the left hand sides of these equations are singularity-free at the transition we must require

$$\dot{A}_i^{(B)}(t_{BR},\vec{x}) = \dot{A}_i^{(R)}(t_{BR},\vec{x}). \tag{3.44}$$

Our simple model of the inflation to radiation dominance transition does not include the physics responsible for the homogeneous part of Φ developing a constant vacuum expectation value; to phenomenologically model this process we shall assume that

$$\frac{1}{\bar{e}^2}\, f\left(\Phi_b(t_{R\Phi})\right) = \frac{1}{e^2}, \tag{3.45}$$

where $\bar{e}$ and e are the Maxwell coupling constants before and after the transition. We also assume that $f(\Phi_b)$ does not jump at this transition and that the potential of the scalar field after the transition is chosen to ensure that the experimental and observational constraints on the scalar field coupling to the vector potential are satisfied.

Now eqns. (3.39) and (3.40) may be written as

$$\frac{d}{dt}\left[a\, f(\Phi_b)\, \dot{A}_1\right] = 0, \tag{3.46}$$

$$\frac{d}{dt}\left[a f(\Phi_b)\, \dot{A}_I\right] - \frac{1}{a}\, f(\Phi_b)\, \partial_1^2 A_I = 0. \tag{3.47}$$

At the transition we define

$$\frac{d}{dt}\left[a f(\Phi_b)\dot{A}_i\right] = \lim_{\Delta t\to 0}\frac{1}{2\Delta t}\left[\left\{a f(\Phi_b)\dot{A}_i\right\}^{(R)}(t_{R\Phi}+\Delta t) - \left\{a f(\Phi_b)\dot{A}_i\right\}^{(\Phi)}(t_{R\Phi}-\Delta t)\right], \tag{3.48}$$

where the superscripts (R) and (Φ) signify that the quantities in the curly brackets are to be evaluated in the radiation and inflation epochs, at the appropriate time. Expanding the terms on the right hand side of this equation and making use of the fact that the scale factor and $f(\Phi_b)$ do not jump at the transition, we have

$$\frac{d}{dt}\left[a\, f(\Phi_b)\, \dot{A}_i\right] = a(t_{R\Phi})\, f(\Phi_b(t_{R\Phi}))\left[\lim_{\Delta t\to 0}\frac{1}{2\Delta t}\left\{\left[\dot{A}_i^{(R)}(t_{R\Phi}) - \dot{A}_i^{(\Phi)}(t_{R\Phi})\right] + \cdots\right\}\right], \tag{3.49}$$

where the ellipses denote terms that are of order Δt or smaller. The second term on the left hand side of eqn. (3.47) is singularity-free at the transition and if the left hand sides of eqns. (3.46) and (3.47) are to be singularity-free at the transition we must require

$$\dot{A}_i^{(R)}(t_{R\Phi}, \vec{x}) = \dot{A}_i^{(\Phi)}(t_{R\Phi}, \vec{x}). \tag{3.50}$$

Finally, we note, from eqns. ($B19$) and ($B20$) of Appendix B, that these joining conditions for A_i and $\dot{A}_i$ ensure that the electric and magnetic fields do not jump at the transitions.

3.4 Determining the Constants of Integration

From eqns. (2.33), (2.34), (2.49), (3.23), (3.24), (3.42) and (3.50) we find that the constants in the radiation epoch vector potential solution, eqns. (3.23) and (3.24), are

$$\begin{aligned} C_1^{(R)} =& C_1, && (3.51)\\ C_{I\pm}^{(R)} =& \pi\,\frac{\sqrt{3}}{4}\,\lambda_R\, e^{x_R/4} e^{-(2-q)\hat{\nu}\bar{x}_R/q} H_{1/2}^{\binom{2}{1}}\left(\sqrt{3}\lambda_R\, e^{x_R/2}\right) \\ &\times\left[H_{\hat{\nu}}^{(1)}\left(\lambda\, e^{-(2-q)\bar{x}_R/q}\right) \pm i\, H_{\hat{\nu}-1}^{(1)}\left(\lambda\, e^{-(2-q)\bar{x}_R/q}\right)\right] C_{I+}, && (3.52)\end{aligned}$$

where C_1 is defined in eqn. (2.33), C_{I+} in eqns. (2.34), (2.47) and (2.49), λ and $\hat{\nu}$ in eqns. (2.32) and (2.35), and λ_R in eqn. (3.26); eqn. (3.25) defines

$$e^{x_R} = (M_R)^{-1}, \tag{3.53}$$

and we have defined

$$e^{\bar{x}_R} = \frac{4}{q}\, e^{x_R}. \tag{3.54}$$

We note that when $\hat{\nu} = 1/2$ ($\alpha = 0$) the second term in the square parentheses in eqn. (3.52) $\pm i H_{-1/2}^{(1)} = \mp H_{1/2}^{(1)}$, so in this case $C_{I+}^{(R)} = 0$.

Similarly, we find that the baryon epoch constants in eqns. (3.27) and (3.28) are

$$C_1^{(B)} = C_1^{(R)}, \tag{3.55}$$

$$C_{I\pm}^{(B)} = \frac{\pi}{2} \widetilde{\lambda}_B \, e^{x_B/6} e^{\bar{x}_B/4} H_{1/2}^{\binom{1}{2}} \left(\sqrt{3}\lambda_R \, e^{\bar{x}_B/2} \right) H_{1/2}^{\binom{2}{1}} \left(\widetilde{\lambda}_B \, e^{x_B/3} \right) C_{I\pm}^{(R)}, \tag{3.56}$$

where $\widetilde{\lambda}_B$ is defined in eqn. (3.30), eqn. (3.29) defines

$$e^{x_B} = (M_B)^{-1}, \tag{3.57}$$

and we have defined

$$e^{\bar{x}_B} = \frac{3}{4} e^{x_B}. \tag{3.58}$$

Since $C_{I+}^{(R)} = 0$ when $\hat{\nu} = 1/2$, eqn. (3.56) implies $C_{I+}^{(B)} = 0$ in this case.

Of particular interest are those perturbations that were on scales larger than the Hubble radius at the transitions (since they are insensitive to the sub Hubble scale physics which we have ignored). Expanding (3.52) we find that the leading large-scale contribution is

$$\begin{aligned} C_{I\pm}^{(R)} = \pm \left(\frac{16\pi}{m_p^2} \right)^{1/2} \bar{e} \, \frac{\pi}{2^{9/4}} \left[(2-q) \left(\frac{q}{2} \right)^{(2-q)/q} \right]^{\hat{\nu}} \left(\frac{q}{2-q} \right)^{-\alpha\sqrt{(q/2)}} \\ \times \frac{\csc(\hat{\nu}\pi)}{\Gamma(1-\hat{\nu})} \, e^{i(\hat{\nu}+1/2)\pi/2} \left[(a_0)^{\sqrt{(q/2)}} e^{-\Phi^{(0)}/2} \right]^{\alpha} \left(\frac{k}{\lambda} \right)^{-(1+\alpha\sqrt{2q})/2} \\ \times (H_{R\Phi})^{[1+4(2-q)\hat{\nu}/q]/4} \left(\frac{k}{a_{R\Phi} H_{R\Phi}} \right)^{(1/2-\hat{\nu})} \\ \times \left[1 \pm \frac{e^{-i(\hat{\nu}+1/2)\pi}}{(2-q)^{(2\hat{\nu}-1)}} \frac{\Gamma(1-\hat{\nu})}{\Gamma(\hat{\nu})} \left(\frac{k}{a_{R\Phi} H_{R\Phi}} \right)^{(2\hat{\nu}-1)} + \cdots \right], \end{aligned} \tag{3.59}$$

where $a_{R\Phi}$ and $H_{R\Phi}$ are the scale factor and Hubble parameter at the reheating transition, the ellipses denote terms down by positive powers of $k/(a_{R\Phi} H_{R\Phi})$, and the first term in the square parentheses dominates when $\hat{\nu} > 1/2$ while the second dominates when $\hat{\nu} < 1/2$; again, when $\hat{\nu} = 1/2$ $C_{I+}^{(R)} = 0$. Similarly, expanding eqn. (3.56), and using eqn. (3.59),

$$C_{I\pm}^{(B)} = \pm \left(\frac{16\pi}{m_p^2} \right)^{1/2} \bar{e} \, \frac{\pi}{4} \left(\frac{3}{2} \right)^{1/6} \left[(2-q) \left(\frac{q}{2} \right)^{(2-q)/q} \right]^{\hat{\nu}} \left(\frac{q}{2-q} \right)^{-\alpha\sqrt{(q/2)}}$$

$$
\begin{aligned}
&\times \frac{\csc(\hat{\nu}\pi)}{\Gamma(1-\hat{\nu})}\, e^{i(\hat{\nu}+1/2)\pi/2} \left[(a_0)^{\sqrt{(q/2)}} e^{-\Phi^{(0)}/2} \right]^{\alpha} \left(\frac{k}{\lambda} \right)^{-(1+\alpha\sqrt{2q})/2} \\
&\times (H_{R\Phi})^{[1+4(2-q)\hat{\nu}/q]/4} (H_{BR})^{-1/12} \left(\frac{k}{a_{R\Phi} H_{R\Phi}} \right)^{(1/2-\hat{\nu})} \\
&\times \left[1 \pm \frac{e^{-i(\hat{\nu}+1/2)\pi}}{(2-q)^{(2\hat{\nu}-1)}} \frac{\Gamma(1-\hat{\nu})}{\Gamma(\hat{\nu})} \left(\frac{k}{a_{R\Phi} H_{R\Phi}} \right)^{(2\hat{\nu}-1)} + \cdots \right],
\end{aligned} \tag{3.60}
$$

where H_{BR} is the value of the Hubble parameter at the radiation-baryon transition and again the first term in the square parentheses dominates for $\hat{\nu} > 1/2$, the second for $\hat{\nu} < 1/2$, and $C^{(B)}_{I+} = 0$ when $\hat{\nu} = 1/2$.

3.5 Derived Fields

In this section we determine expressions for the Maxwell field-strength tensor and for the energy density carried by the vector potential perturbation during radiation and baryon dominance. These results should be compared to those of Secs. 4.5 and 5 (where we derive equivalent expressions for the nonzero conductivity case).

During radiation dominance, when $\hat{\nu} > 1/2$ we find, for vector potential perturbations that reenter the Hubble radius in this epoch,

$$
\langle F_{0I}(k,a) F^{*}_{0I}(k,a) \rangle = B_1\, k^{2(1-\hat{\nu})}\, a^{-2} \sin^2\left(\frac{k}{aH} \right) + \cdots, \tag{3.61}
$$

$$
\langle F_{1I}(k,a) F^{*}_{1I}(k,a) \rangle = B_1\, k^{2(1-\hat{\nu})} \cos^2\left(\frac{k}{aH} \right) + \cdots, \tag{3.62}
$$

where the ellipses denote terms that are down by positive powers of $k/(a_{R\Phi} H_{R\Phi})$ and

$$
\begin{aligned}
B_1 =& \frac{16\pi}{m_p^2}\, \bar{e}^2\, \frac{\pi}{\sqrt{24}} \left[(2-q) \left(\frac{q}{2} \right)^{(2-q)/q} \right]^{2\hat{\nu}} \left(\frac{q}{2-q} \right)^{-\alpha\sqrt{2q}} \\
&\times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1-\hat{\nu})} \left[(a_0)^{\sqrt{2q}} e^{-\Phi^{(0)}} \right]^{\alpha} \left(\frac{k}{\lambda} \right)^{-(1+\alpha\sqrt{2q})} \left(\frac{k}{\lambda_R} \right) \\
&\times (H_{R\Phi})^{[1+4(2-q)\hat{\nu}/q]/2} \left[a_{R\Phi} H_{R\Phi} \right]^{(2\hat{\nu}-1)}
\end{aligned} \tag{3.63}
$$

[there is no implicit summation over the index I on the left hand sides of eqns. (3.61) and

(3.62) and in similar equations that follow]. Similarly, when $\hat{\nu} < 1/2$ we have

$$\langle F_{0I}(k,a)F^*_{0I}(k,a)\rangle = B_2\, k^{2\hat{\nu}}\, a^{-2} \cos^2\left(\frac{k}{aH}\right) + \cdots, \tag{3.64}$$

$$\langle F_{1I}(k,a)F^*_{1I}(k,a)\rangle = B_2\, k^{2\hat{\nu}} \sin^2\left(\frac{k}{aH}\right) + \cdots, \tag{3.65}$$

where we have defined

$$\begin{aligned} B_2 =& \frac{16\pi}{m_p^2}\, \bar{e}^2\, \frac{\pi}{\sqrt{24}} \left[\frac{1}{(2-q)} \left(\frac{q}{2}\right)^{(2-q)/q}\right]^{2\hat{\nu}} (2-q)^2 \left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \\ &\times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(\hat{\nu})} \left[(a_0)^{\sqrt{2q}} e^{-\Phi^{(0)}}\right]^{\alpha} \left(\frac{k}{\lambda}\right)^{-(1+\alpha\sqrt{2q})} \left(\frac{k}{\lambda_R}\right) \\ &\times (H_{R\Phi})^{[1+4(2-q)\hat{\nu}/q]/2} \left[a_{R\Phi} H_{R\Phi}\right]^{(1-2\hat{\nu})}. \end{aligned} \tag{3.66}$$

The results for the $\hat{\nu} = 1/2$ case are tabulated below, eqns. (3.81) and (3.82). We emphasize that the spatial momentum dependence of eqns. (3.61), (3.62), (3.64) and (3.65) is α dependent. We note, from eqn. ($B19$), that the coordinate electric field two-point function $\langle E_I(k,a)E^*_I(k,a)\rangle = \langle F_{0I}(k,a)F^*_{0I}(k,a)\rangle$, while, from eqn. ($B20$), one of the coordinate magnetic field two-point functions $\langle B_2(k,a)B^*_2(k,a)\rangle = a^{-2}\,\langle F_{13}(k,a)F^*_{13}(k,a)\rangle$.

On scales larger than the Hubble radius during radiation dominance the $\hat{\nu} > 1/2$ correlation functions, eqns. (3.61) and (3.62), reduce to the time-independent expressions

$$\langle F_{0I}(k,a)F^*_{0I}(k,a)\rangle = \frac{9}{4}\, B_1 \left(\frac{k}{\lambda_R}\right)^{-4} k^{2(2-\hat{\nu})} + \cdots, \tag{3.67}$$

$$\langle F_{1I}(k,a)F^*_{1I}(k,a)\rangle = B_1\, k^{2(1-\hat{\nu})} + \cdots, \tag{3.68}$$

where the ellipses now also denote terms that are down by positive powers of $k/(aH)$. Similarly, the $\hat{\nu} < 1/2$ expressions, eqns. (3.64) and (3.65), become

$$\langle F_{0I}(k,a)F^*_{0I}(k,a)\rangle = B_2\, k^{2\hat{\nu}}\, a^{-2} + \cdots, \tag{3.69}$$

$$\langle F_{1I}(k,a)F^*_{1I}(k,a)\rangle = \frac{9}{4}\, B_2 \left(\frac{k}{\lambda_R}\right)^{-4} k^{2(1+\hat{\nu})}\, a^2 + \cdots. \tag{3.70}$$

During baryon dominance, for vector potential perturbations that first reenter the Hubble radius during this epoch, we have, for $\hat{\nu} > 1/2$,

$$\langle F_{0I}(k,a) F^*_{0I}(k,a)\rangle = B_3\, k^{2(1-\hat{\nu})}\, a^{-2} \sin^2\left(\frac{2k}{aH}\right) + \cdots , \tag{3.71}$$

$$\langle F_{1I}(k,a) F^*_{1I}(k,a)\rangle = B_3\, k^{2(1-\hat{\nu})} \cos^2\left(\frac{2k}{aH}\right) + \cdots , \tag{3.72}$$

where the ellipses denote terms that are down by positive powers of $k/(a_{R\Phi} H_{R\Phi})$ and $k/(a_{BR} H_{BR})$ and

$$\begin{aligned} B_3 =& \frac{16\pi}{m_p^2}\, \bar{e}^2\, \frac{\pi}{2} \left(\frac{3}{2}\right)^{1/3} \left[(2-q) \left(\frac{q}{2}\right)^{(2-q)/q}\right]^{2\hat{\nu}} \left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \\ & \times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1-\hat{\nu})} \left[(a_0)^{\sqrt{2q}} e^{-\Phi^{(0)}}\right]^{\alpha} \left(\frac{k}{\lambda}\right)^{-(1+\alpha\sqrt{2q})} \left(\frac{k}{\widetilde{\lambda}_B}\right) \\ & \times (H_{R\Phi})^{[1+4(2-q)\hat{\nu}/q]/2} (H_{BR})^{-1/6} \left[a_{R\Phi} H_{R\Phi}\right]^{(2\hat{\nu}-1)} . \end{aligned} \tag{3.73}$$

When $\hat{\nu} < 1/2$ we have

$$\langle F_{0I}(k,a) F^*_{0I}(k,a)\rangle = B_4\, k^{2\hat{\nu}}\, a^{-2} \cos^2\left(\frac{2k}{aH}\right) + \cdots , \tag{3.74}$$

$$\langle F_{1I}(k,a) F^*_{1I}(k,a)\rangle = B_4\, k^{2\hat{\nu}} \sin^2\left(\frac{2k}{aH}\right) + \cdots , \tag{3.75}$$

where we have defined

$$\begin{aligned} B_4 =& \frac{16\pi}{m_p^2}\, \bar{e}^2\, \frac{\pi}{2} \left(\frac{3}{2}\right)^{1/3} \left[\frac{1}{(2-q)} \left(\frac{q}{2}\right)^{(2-q)/q}\right]^{2\hat{\nu}} (2-q)^2 \left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \\ & \times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(\hat{\nu})} \left[(a_0)^{\sqrt{2q}} e^{-\Phi^{(0)}}\right]^{\alpha} \left(\frac{k}{\lambda}\right)^{-(1+\alpha\sqrt{2q})} \left(\frac{k}{\widetilde{\lambda}_B}\right) \\ & \times (H_{R\Phi})^{[1+4(2-q)\hat{\nu}/q]/2} (H_{BR})^{-1/6} \left[a_{R\Phi} H_{R\Phi}\right]^{(1-2\hat{\nu})} . \end{aligned} \tag{3.76}$$

The results for the $\hat{\nu} = 1/2$ case are tabulated below, (3.81) and (3.82). We emphasize that the spatial momentum dependence of (3.71), (3.72), (3.74) and (3.75) is α dependent.

On scales larger than the Hubble radius during baryon domination the $\hat{\nu} > 1/2$ correlation functions, eqns. (3.71) and (3.72), reduce to

$$\langle F_{0I}(k,a) F^*_{0I}(k,a)\rangle = \frac{1}{3}\, B_3 \left(\frac{k}{\widetilde{\lambda}_B}\right)^{-3} k^{2(2-\hat{\nu})}\, a^{-1} + \cdots , \tag{3.77}$$

$$\langle F_{1I}(k,a) F^*_{1I}(k,a)\rangle = B_3\, k^{2(1-\hat\nu)} + \cdots, \tag{3.78}$$

while the $\hat\nu < 1/2$ expressions, eqns. (3.74) and (3.75), become

$$\langle F_{0I}(k,a) F^*_{0I}(k,a)\rangle = B_4\, k^{2\hat\nu}\, a^{-2} + \cdots, \tag{3.79}$$

$$\langle F_{1I}(k,a) F^*_{1I}(k,a)\rangle = \frac{1}{3}\, B_4 \left(\frac{k}{\widetilde\lambda_B}\right)^{-3} k^{2(1+\hat\nu)}\, a + \cdots. \tag{3.80}$$

When $\hat\nu = 1/2$ we find, during both radiation and baryon dominance,

$$\langle F_{0I}(k,a) F^*_{0I}(k,a)\rangle = \frac{8\pi}{m_p^2}\, \bar e^2\, \frac{k}{a^2}, \tag{3.81}$$

$$\langle F_{1I}(k,a) F^*_{1I}(k,a)\rangle = \frac{8\pi}{m_p^2}\, \bar e^2\, k. \tag{3.82}$$

During radiation and baryon dominance the energy density spectrum of the I^{th} component of the vector potential perturbation is

$$\epsilon_I(k,t) = \frac{m_p^2}{8\pi}\, \frac{1}{e^2 a^2} \left[\langle F_{0I}(k,t) F^*_{0I}(k,t)\rangle + \frac{1}{a^2}\, \langle F_{1I}(k,t) F^*_{1I}(k,t)\rangle \right]. \tag{3.83}$$

During radiation dominance, for vector potential perturbations that reenter the Hubble radius during this epoch, we have, from eqns. (3.61) and (3.62), for $\hat\nu > 1/2$,

$$\epsilon_I(k,t) = \frac{m_p^2}{8\pi}\, \frac{1}{e^2}\, B_1\, k^{2(1-\hat\nu)}\, a^{-4} + \cdots, \tag{3.84}$$

while for $\hat\nu < 1/2$ we have, from eqns. (3.64) and (3.65),

$$\epsilon_I(k,t) = \frac{m_p^2}{8\pi}\, \frac{1}{e^2}\, B_2\, k^{2\hat\nu}\, a^{-4} + \cdots. \tag{3.85}$$

Similarly, during baryon dominance, we find, for $\hat\nu > 1/2$,

$$\epsilon_I(k,t) = \frac{m_p^2}{8\pi}\, \frac{1}{e^2}\, B_3\, k^{2(1-\hat\nu)}\, a^{-4} + \cdots, \tag{3.86}$$

while for $\hat\nu < 1/2$ we have

$$\epsilon_I(k,t) = \frac{m_p^2}{8\pi}\, \frac{1}{e^2}\, B_4\, k^{2\hat\nu}\, a^{-4} + \cdots. \tag{3.87}$$

We emphasize that the spatial momentum dependence of eqns. (3.84) – (3.87) is α dependent. When $\hat\nu = 1/2$ the energy density carried by vector potential perturbations that reenter the

Hubble radius during either radiation dominance or baryon dominance is

$$\epsilon_I(k,t) = 2\,\frac{k}{a^4}; \tag{3.88}$$

it is pleasing to note that this agrees with eqn. (2.68).

4. Radiation Dominance: Nonzero Conductivity

In this section we study the evolution of vector potential irregularities in the radiation dominated epoch, keeping track of the effects of the electromagnetic coupling between these perturbations and charged particles by using a simplified Ohm's law.

4.1 Spatial Irregularities

In this section we tabulate the solutions of the electric field and magnetic field perturbation equations of motion, eqns. $(B26)$ – $(B31)$, during radiation dominance.

Working in spatial Fourier space and choosing the x^1 axis to lie along the spatial momentum vector $\vec{k}$ of the perturbation of interest, we find that the longitudinal coordinate electric and magnetic fields vanish,

$$E_1^{(R)}(t) = 0, \tag{4.1}$$

$$B_1^{(R)}(t) = 0; \tag{4.2}$$

it may be verified that these solutions satisfy the $i = 1$ components of eqns. $(B28)$ – $(B31)$.

Replacing the independent variable t by the variable x defined in eqn. (3.25) we find that the transverse parts of the wave equations $(B30)$ and $(B31)$ reduce to

$$\mathcal{E}_I'' - \left[\frac{1}{2} - 4\pi\sigma e^x\right]\mathcal{E}_I' + \left[2\pi\sigma + \frac{3}{4}\,\lambda_R^2\right] e^x\,\mathcal{E}_I = 0, \tag{4.3}$$

$$\mathcal{B}_I'' - \left[\frac{1}{2} - 4\pi\sigma e^x\right]\mathcal{B}_I' + \frac{3}{4}\,\lambda_R^2\, e^x\,\mathcal{B}_I = 0, \tag{4.4}$$

where $\mathcal{E}_I$ and $\mathcal{B}_I$ are the transverse coordinate electric and magnetic fields multiplied by a, $(B32)$ and $(B33)$, λ_R is defined in eqn. (3.26) and σ is the conductivity [$(A61)$ and $(B25)$].

The solution of eqn. (4.3) is given by

$$\mathcal{E}_I^{(R)}(x) = \hat{C}_{I+}^{(R)}\, R^{(1)}\left(f+\frac{1}{2},\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right) + \hat{C}_{I-}^{(R)}\, R^{(2)}\left(f+\frac{1}{2},\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right), \quad (4.5)$$

where $\hat{C}_{I\pm}^{(R)}$ are k dependent constants of integration corresponding to general coordinate invariant and $U(1)$ gauge invariant solutions,

$$f = \frac{3}{16\pi}\,\frac{\lambda_R^2}{\sigma} \quad (4.6)$$

[which should not be confused with the f of eqns. (2.1) and (2.21)], and we have defined

$$R^{(1)}\left(f+\frac{1}{2},\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right) = \frac{1}{\pi}\,\Gamma(f+1)\, U\left(f+\frac{1}{2},\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right), \quad (4.7)$$

$$\begin{aligned} R^{(2)}\left(f+\frac{1}{2},\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right) = {} & \frac{2}{\sqrt{\pi}}\, M\left(f+\frac{1}{2},\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right) \quad (4.8)\\ & -\frac{1}{\pi}\,\Gamma(f+1)\, U\left(f+\frac{1}{2},\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right), \end{aligned}$$

where Γ is the gamma function and M and U are confluent hypergeometric functions, [79]. We note that only the first argument of $R^{(1)}$ and $R^{(2)}$ is momentum dependent while only the third argument of these functions is time dependent. The solution of eqn. (4.4) is

$$\mathcal{B}_I^{(R)}(x) = \hat{D}_{I+}^{(R)}\, R^{(1)}\left(f,\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right) + \hat{D}_{I-}^{(R)}\, R^{(2)}\left(f,\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right), \quad (4.9)$$

where $\hat{D}_{I\pm}^{(R)}$ are k dependent constants of integration corresponding to general coordinate invariant and $U(1)$ gauge invariant solutions, and we have defined

$$R^{(1)}\left(f,\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right) = \frac{1}{\pi}\,\Gamma\left(f+\frac{1}{2}\right) U\left(f,\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right), \quad (4.10)$$

$$\begin{aligned} R^{(2)}\left(f,\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right) = {} & \frac{2}{\sqrt{\pi}}\, M\left(f,\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right) \quad (4.11)\\ & -\frac{1}{\pi}\,\Gamma\left(f+\frac{1}{2}\right) U\left(f,\frac{1}{2};-f^{-1}\frac{3}{4}\,\lambda_R^2\, e^x\right). \end{aligned}$$

Once again, we note that only the first argument of $R^{(1)}$ and $R^{(2)}$ is spatial momentum dependent while only the third argument of these functions is time dependent.

Equations (4.5) and (4.9) satisfy the $i = I$ components of eqns. ($B28$) and ($B29$) if

$$\hat{D}^{(R)}_{I\pm} = \mp \frac{1}{\sqrt{f}} \frac{\Gamma(f+1)}{\Gamma(f+1/2)} \, \epsilon_{1IJ} \, \hat{C}^{(R)}_{J\pm}, \tag{4.12}$$

where ϵ_{ijk} is the totally antisymmetric tensor density ($\epsilon_{123} = 1$).

4.2 Zero- and Perfect-Conductivity Limits and Asymptotics

It is useful to note, from eqns. (3.17), (3.25), (3.26) and (4.6), that

$$-f^{-1} \frac{3}{4} \lambda_R^2 \, e^x = -2\pi \frac{\sigma}{H}, \tag{4.13}$$

$$f = \frac{1}{8\pi} \left(\frac{k}{aH} \right)^2 \frac{H}{\sigma}, \tag{4.14}$$

$$\sqrt{3} \, \lambda_R \, e^{x/2} = \frac{k}{aH} = k \int^t \frac{dt'}{a(t')} = k\tau, \tag{4.15}$$

where τ is conformal time.

In the zero-conductivity limit, $\sigma \to 0$ (or $f \to \infty$), when the electromagnetic coupling between the vector potential perturbation and the electrically conducting fluid is neglected, we find, from eqns. (4.5) – (4.12),

$$\begin{aligned} E^{(R)}_I(x) = -i \left(\frac{\sqrt{3}}{2} \right)^{3/2} \frac{\lambda_R^{3/2}}{k} \, e^{-x/4} \\ \times \left[\hat{C}^{(R)}_{I+} \, H^{(2)}_{1/2} \left(\sqrt{3} \, \lambda_R \, e^{x/2} \right) - \hat{C}^{(R)}_{I-} \, H^{(1)}_{1/2} \left(\sqrt{3} \, \lambda_R \, e^{x/2} \right) \right], \end{aligned} \tag{4.16}$$

$$\begin{aligned} B^{(R)}_I(x) = i \left(\frac{\sqrt{3}}{2} \right)^{3/2} \frac{\lambda_R^{3/2}}{k} \, e^{-x/4} \, \epsilon_{1IJ} \\ \times \left[\hat{C}^{(R)}_{J+} \, H^{(2)}_{1/2} \left(\sqrt{3} \, \lambda_R \, e^{x/2} \right) + \hat{C}^{(R)}_{J-} \, H^{(1)}_{1/2} \left(\sqrt{3} \, \lambda_R \, e^{x/2} \right) \right]. \end{aligned} \tag{4.17}$$

These expressions and eqns. (4.1) and (4.2) coincide with those derived from eqns. (3.23) and (3.24) [by using eqns. ($B19$) and ($B20$)], provided we relate the constants of integration in eqn. (3.24), $C^{(R)}_{I\pm}$, to $\hat{C}^{(R)}_{I\pm}$ through

$$\hat{C}^{(R)}_{I\mp} = -\left(\frac{2}{\sqrt{3}}\right)^{1/2} \frac{k}{\lambda_R^{1/2}}\, C^{(R)}_{I\pm}. \tag{4.18}$$

We note that in the $k \to 0$ large-scale limit $f \to 0$ [eqn. (4.14)], which clashes with the approximation, that $f \to \infty$, used to derive eqns. (4.16) and (4.17).

In the perfect-conductivity limit, $\sigma \to \infty$ (or $f \to 0$), we find

$$E^{(R)}_I(x) = -\frac{i}{\pi}\, f^{1/2}\, k^{-1} e^{-x} \left[\hat{C}^{(R)}_{I+} - \hat{C}^{(R)}_{I-}\right] + \cdots, \tag{4.19}$$

$$B^{(R)}_I(x) = -\frac{\sqrt{3}}{2\pi}\, f^{-1/2}\, \frac{\lambda_R}{k}\, e^{-x/2}\, \epsilon_{1IJ} \left[\hat{C}^{(R)}_{J+} - \hat{C}^{(R)}_{J-}\right] + \cdots, \tag{4.20}$$

where the ellipses denote terms that are suppressed by positive powers of f. If we make the usual perfect-conductivity MHD approximation that $E^{(R)}_i = 0$, eqn. $(B29)$ implies

$$B^{(R)}_I \propto a^{-1} \propto e^{-x/2}, \tag{4.21}$$

which agrees with the time dependence of eqn. (4.20). We note that in the $k \to \infty$ small-scale limit $f \to \infty$ [eqn. (4.14)], which clashes with the $f \to 0$ approximation used to derive eqns. (4.19) and (4.20).

In the small-scale, $k \to \infty$, asymptotic limit we find the expected spatial momentum supported oscillatory behaviour,

$$aE^{(R)}_I(k,t) = \frac{1}{\sqrt{\pi}}\, e^{-\pi\sigma/H} \left[\hat{C}^{(R)}_{I+}\, e^{-ik\tau} + \hat{C}^{(R)}_{I-}\, e^{ik\tau}\right], \tag{4.22}$$

$$aB^{(R)}_I(k,t) = -\frac{1}{\sqrt{\pi}}\, e^{-\pi\sigma/H}\, \epsilon_{1IJ} \left[\hat{C}^{(R)}_{J+}\, e^{-ik\tau} - \hat{C}^{(R)}_{J-}\, e^{ik\tau}\right]; \tag{4.23}$$

except for the σ dependent factor these expressions agree with eqns. (4.16) and (4.17), so the zero-conductivity limit of these expressions coincide with (the small-scale limit of) eqns. (4.16) and (4.17). Eqns. (4.22) and (4.23) were derived by taking $f \to \infty$, eqn. (4.14), so the perfect-conductivity, $\sigma \to \infty$ (or $f \to 0$), limit of these equations need not agree with the small-scale limit of eqns. (4.19) and (4.20) (which were derived by taking $f \to 0$).

In the large-scale, $k \to 0$, asymptotic limit we have

$$aE_I^{(R)}(k,t) = \frac{1}{\pi}\, e^{-2\pi\sigma/H}\left[2\sqrt{\pi}\,\hat{C}_{I-}^{(R)} + \left(\hat{C}_{I+}^{(R)} - \hat{C}_{I-}^{(R)}\right)\Gamma\left(\frac{1}{2}, -2\pi\,\frac{\sigma}{H}\right) + \cdots\right], \tag{4.24}$$

$$\begin{aligned} aB_I^{(R)}(k,t) = &-\frac{1}{\pi}\,\epsilon_{1IJ}\Big[f^{-1/2}\left(\hat{C}_{J+}^{(R)} - \hat{C}_{J-}^{(R)}\right) \\ &- i f^{1/2}\left\{\pi - \sqrt{\pi}\,\Gamma\left(\frac{1}{2}, 2\pi\,\frac{\sigma}{H}\right)\right\}\left(\hat{C}_{J+}^{(R)} + \hat{C}_{J-}^{(R)}\right) + \cdots\Big], \end{aligned} \tag{4.25}$$

where the ellipses denote terms down by positive powers of k and Γ represents the incomplete gamma function. It may be shown that in the perfect-conductivity, $\sigma \to \infty$, limit these expressions agree with eqns. (4.19) and (4.20). The zero-conductivity, $\sigma \to 0$ (or $f \to \infty$), limit clashes with the assumption, $f \to 0$, used to derive eqns. (4.24) and (4.25).

4.3 Transition Model and Derivation of Joining Conditions

The transition model we use is similar to that used in Sec. 3.3 except we now also assume that at the inflation-radiation transition (at $t = t_{R\Phi}$) the conductivity σ jumps from a vanishingly small value to a large value.

To derive joining conditions for the electric and magnetic fields at the transition we require that the equations of motion remain singularity-free at the transition. It proves convenient to consider the vector potential equations, eqns. (2.18), (2.19), $(B34)$ and $(B35)$,

$$\ddot{A}_1 + \left(\frac{\dot{a}}{a} + \dot{\Phi}_b\frac{f'(\Phi_b)}{f(\Phi_b)} + 4\pi\sigma\right)\dot{A}_1 = 0, \tag{4.26}$$

$$\ddot{A}_I + \left(\frac{\dot{a}}{a} + \dot{\Phi}_b\frac{f'(\Phi_b)}{f(\Phi_b)} + 4\pi\sigma\right)\dot{A}_I - \frac{1}{a^2}\partial_1^2 A_I = 0, \tag{4.27}$$

$$\partial_1 \dot{A}_1 = 0, \tag{4.28}$$

where, as in Sec. 3.4, $f'(\Phi_b) = 0$ during radiation dominance (for $t > t_{R\Phi}$) and we take $\sigma = 0$ during inflation (when $t < t_{R\Phi}$).

Following the analysis of Sec. 3.3 we rewrite eqns. (4.26) and (4.27),

$$\frac{d}{dt}\left[a\, f(\Phi_b)\, e^{4\pi\sigma t}\dot{A}_1\right] = 0, \tag{4.29}$$

$$\frac{d}{dt}\left[a f(\Phi_b)\, e^{4\pi\sigma t} \dot{A}_I\right] - \frac{1}{a} f(\Phi_b)\, e^{4\pi\sigma t} \partial_1^2 A_I = 0. \tag{4.30}$$

Since a and $f(\Phi_b)$ do not jump at the reheating transition, the joining conditions are

$$A_i^{(R)}(t_{R\Phi}, \vec{x}) = A_i^{(\Phi)}(t_{R\Phi}, \vec{x}), \tag{4.31}$$

$$\dot{A}_i^{(R)}(t_{R\Phi}, \vec{x}) = e^{-4\pi\sigma t_{R\Phi}}\, \dot{A}_i^{(\Phi)}(t_{R\Phi}, \vec{x}), \tag{4.32}$$

or in terms of the coordinate electric and magnetic fields, eqns. (B19) and (B20),

$$E_i^{(R)}(t_{R\Phi}, \vec{x}) = e^{-4\pi\sigma t_{R\Phi}}\, E_i^{(\Phi)}(t_{R\Phi}, \vec{x}), \tag{4.33}$$

$$B_i^{(R)}(t_{R\Phi}, \vec{x}) = B_i^{(\Phi)}(t_{R\Phi}, \vec{x}). \tag{4.34}$$

So for a large enough conductivity during reheating it would seem that the electric field accelerates charged particles and is dissipated while the magnetic field is not affected by the transition. Presumably the energy fed into the plasma by the electric field will not be significant compared to that fed in by the reheating process itself.

4.4 Determining the Constants of Integration

During inflation, we find, from eqns. (2.33), (2.34), (B19) and (B20), that the coordinate electric field and coordinate magnetic field perturbations are given by

$$E_1^{(\Phi)}(\bar{x}) = 0, \tag{4.35}$$

$$E_I^{(\Phi)}(\bar{x}) = \frac{2-q}{q}\, C_{I+}\, \lambda\, e^{-[1+(2-q)(\hat{\nu}+1)/q]\bar{x}}\, H_{\hat{\nu}-1}^{(1)}\left(\lambda\, e^{-(2-q)\bar{x}/q}\right), \tag{4.36}$$

$$B_1^{(\Phi)}(\bar{x}) = 0, \tag{4.37}$$

$$\epsilon_{1IJ}\, B_J^{(\Phi)}(\bar{x}) = i\, \frac{2-q}{q}\, C_{I+}\, \lambda\, e^{-[1+(2-q)(\hat{\nu}+1)/q]\bar{x}}\, H_{\hat{\nu}}^{(1)}\left(\lambda\, e^{-(2-q)\bar{x}/q}\right), \tag{4.38}$$

where we have used eqn. (2.49), C_{I+} is defined in eqns. (2.47) and (2.49), λ is given in eqn. (2.32), $\hat{\nu}$ is defined in eqn. (2.35), and $\bar{x}$ represents the variable x of eqn. (2.28).

Since the longitudinal electric field and longitudinal magnetic field perturbations vanish during inflation, eqns. (4.35) and (4.37), and during radiation dominance, eqns. (4.1) and (4.2), the $i = 1$ components of the joining conditions (4.33) and (4.34) are identically satisfied.

The $i = I$ components of the joining conditions (4.33) and (4.34) determine the radiation epoch constants of integration $\hat{C}^{(R)}_{I\pm}$ in eqns. (4.5), (4.9) and (4.12),

$$\begin{aligned}\hat{C}^{(R)}_{I\pm} =& \frac{\pi}{2}\, a_{R\Phi} \exp\left[f^{-1}\frac{3}{4}\,\lambda_R^2\, e^{x_R}\right] \\ &\times \left[R^{\binom{2}{1}}\left(f, \frac{1}{2}; -f^{-1}\frac{3}{4}\,\lambda_R^2\, e^{x_R}\right) e^{-4\pi\sigma t_{R\Phi}} E^{(\Phi)}_I(\bar{x}_R)\right. \\ &\left.\pm \sqrt{f}\,\frac{\Gamma(f+1/2)}{\Gamma(f+1)}\, R^{\binom{2}{1}}\left(f+\frac{1}{2}, \frac{1}{2}; -f^{-1}\frac{3}{4}\,\lambda_R^2\, e^{x_R}\right) \epsilon_{1IJ}\, B^{(\Phi)}_J(\bar{x}_R)\right],\end{aligned} \tag{4.39}$$

where $R^{(1)}$ and $R^{(2)}$ are defined in (4.7), (4.8), (4.10) and (4.11), x_R and $\bar{x}_R$ in (3.53) and (3.54), and $a_{R\Phi}$ is the scale factor at the reheating transition. In the zero-conductivity limit, when $\sigma \to 0$ (or $f \to \infty$), (4.39) reduce to (3.52) [using (4.18), (4.36) and (4.38)].

In the large-scale, $k \to 0$, limit, we find, on expanding eqns. (4.36), (4.38) and (4.39),

$$\begin{aligned}\hat{C}^{(R)}_{I+} =& i\,\sqrt{\pi}\,\frac{2^{\hat{\nu}-1}}{(2-q)}\csc(\hat{\nu}\pi)\left(\frac{k}{\lambda}\right)^{\hat{\nu}}\frac{1}{a_{R\Phi}H_{R\Phi}}\, C_{I+}\, k^{2-\hat{\nu}} \\ &\times\left[e^{2\pi\sigma/H_{R\Phi}} e^{-4\pi\sigma t_{R\Phi}}\left\{\frac{1}{\Gamma(2-\hat{\nu})} + \frac{e^{-i\hat{\nu}\pi}}{\Gamma(\hat{\nu})(2-q)^{2(\hat{\nu}-1)}}\left(\frac{k}{a_{R\Phi}H_{R\Phi}}\right)^{2(\hat{\nu}-1)} + \cdots\right\}\right. \\ &\quad - i\,\frac{2-q}{2}\left(\frac{H_{R\Phi}}{2\pi\sigma}\right)^{1/2}\left\{2\sqrt{\pi} - \Gamma\left(\frac{1}{2}, -\frac{2\pi\sigma}{H_{R\Phi}}\right)\right\} \\ &\quad\left.\times\left\{\frac{1}{\Gamma(1-\hat{\nu})} - \frac{e^{-i\hat{\nu}\pi}}{\Gamma(1+\hat{\nu})(2-q)^{2\hat{\nu}}}\left(\frac{k}{a_{R\Phi}H_{R\Phi}}\right)^{2\hat{\nu}} + \cdots\right\} + \cdots\right],\end{aligned} \tag{4.40}$$

$$\begin{aligned}\hat{C}^{(R)}_{I-} =& i\,\sqrt{\pi}\,\frac{2^{\hat{\nu}-1}}{(2-q)}\csc(\hat{\nu}\pi)\left(\frac{k}{\lambda}\right)^{\hat{\nu}}\frac{1}{a_{R\Phi}H_{R\Phi}}\, C_{I+}\, k^{2-\hat{\nu}} \\ &\times\left[e^{2\pi\sigma/H_{R\Phi}} e^{-4\pi\sigma t_{R\Phi}}\left\{\frac{1}{\Gamma(2-\hat{\nu})} + \frac{e^{-i\hat{\nu}\pi}}{\Gamma(\hat{\nu})(2-q)^{2(\hat{\nu}-1)}}\left(\frac{k}{a_{R\Phi}H_{R\Phi}}\right)^{2(\hat{\nu}-1)} + \cdots\right\}\right. \\ &\quad + i\,\frac{2-q}{2}\left(\frac{H_{R\Phi}}{2\pi\sigma}\right)^{1/2}\Gamma\left(\frac{1}{2}, -\frac{2\pi\sigma}{H_{R\Phi}}\right) \\ &\quad\left.\times\left\{\frac{1}{\Gamma(1-\hat{\nu})} - \frac{e^{-i\hat{\nu}\pi}}{\Gamma(1+\hat{\nu})(2-q)^{2\hat{\nu}}}\left(\frac{k}{a_{R\Phi}H_{R\Phi}}\right)^{2\hat{\nu}} + \cdots\right\} + \cdots\right].\end{aligned} \tag{4.41}$$

In these equations we have only recorded some of the leading large-scale terms in the inflation epoch expressions for the electric and magnetic fields (evaluated at the transition), and the ellipses denote terms that are down by positive powers of k.

In the large-conductivity limit (which is the limit of interest, [60]) the combination $\hat{C}^{(R)}_{I+} - \hat{C}^{(R)}_{I-}$ is of interest, eqns. (4.19) and (4.20). We have, from eqns. (4.40) and (4.41), for $\hat{\nu} > 0$,

$$\hat{C}^{(R)}_{I+} - \hat{C}^{(R)}_{I-} = \pi\, 2^{\hat{\nu}-1}\, \frac{\csc(\hat{\nu}\pi)}{\Gamma(1-\hat{\nu})} \left(\frac{H_{R\Phi}}{2\pi\sigma}\right)^{1/2} \left(\frac{k}{\lambda}\right)^{\hat{\nu}} \frac{1}{a_{R\Phi} H_{R\Phi}}\, C_{I+}\, k^{2-\hat{\nu}} + \cdots, \tag{4.42}$$

and for $\hat{\nu} < 0$,

$$\begin{aligned}\hat{C}^{(R)}_{I+} - \hat{C}^{(R)}_{I-} =& -\pi\, 2^{\hat{\nu}-1}\, \frac{\csc(\hat{\nu}\pi)}{\Gamma(1+\hat{\nu})}\, \frac{e^{-i\hat{\nu}\pi}}{(2-q)^{2\hat{\nu}}} \left(\frac{H_{R\Phi}}{2\pi\sigma}\right)^{1/2} \\ &\times \left(\frac{k}{\lambda}\right)^{\hat{\nu}} \frac{1}{(a_{R\Phi} H_{R\Phi})^{2\hat{\nu}+1}}\, C_{I+}\, k^{2+\hat{\nu}} + \cdots. \end{aligned}\tag{4.43}$$

We note that the leading terms in these expressions come form the inflation epoch magnetic field term in eqn. (4.39).

4.5 Two-Point Correlation Functions and Magnetic Energy Density in the Large-Conductivity Limit

In this section we tabulate two-point correlation functions for the electric and magnetic fields (as well as an expression for the energy density carried by these perturbations) in the large-conductivity limit. The results of this section should be compared to those of Sec. 3.5 (where we derive equivalent expressions for the zero conductivity model).

From eqns. (2.47), (2.49), (4.19), (4.20) and (4.42), we find, in the large-conductivity ($\sigma \to \infty$) limit, for $\hat{\nu} > 0$,

$$\begin{aligned}\left\langle E^{(R)}_I(k,a)\, E^{(R)*}_I(k,a)\right\rangle =& \frac{1}{\sigma^2}\, \frac{16\pi}{m_p^2}\, \bar{e}^2\, e^{-\alpha\Phi^{(0)}}\, (a_0)^{\alpha\sqrt{2q}}\, \frac{2^{2(\hat{\nu}-3)}}{\pi} \left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \\ &\times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1-\hat{\nu})} \left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)} \frac{k^{2(2-\hat{\nu})}}{a^4} + \cdots, \end{aligned}\tag{4.44}$$

$$\left\langle \epsilon_{1IJ} B_J^{(R)}(k,a)\, \epsilon_{1IK} B_K^{(R)*}(k,a) \right\rangle = \frac{16\pi}{m_p^2}\, \bar{e}^2\, e^{-\alpha\Phi^{(0)}}\, (a_0)^{\alpha\sqrt{2q}}\, \pi\, 2^{2(\hat{\nu}-1)} \left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1-\hat{\nu})} \left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)} \frac{k^{2(1-\hat{\nu})}}{a^2} + \cdots. \tag{4.45}$$

In these and following equations there is no summation over the index I on the left hand sides, $\bar{e}$ is the Maxwell coupling constant during inflation [(2.1)], $\Phi^{(0)}$ is defined in (2.20) and (2.23), α in (2.21), λ in (2.32), $\hat{\nu}$ in (2.35), and the ellipses now also denote terms down by negative powers of the conductivity σ. From (4.44) we note that $\sigma E_I^{(R)}$ [and the transverse electric field part of the current, ($B25$)] remains finite in the perfect-conductivity, $\sigma \to \infty$, limit — as is assumed in the usual perfect-conductivity MHD approximation. Using ($B20$) it may be shown that the large-scale expansion of (3.68) (for the $\sigma = 0$ case, valid when $\hat{\nu} > 1/2$) coincides with (4.45).

When $\hat{\nu} < 0$, we have, on using eqn. (4.43), in the large-conductivity limit,

$$\left\langle E_I^{(R)}(k,a)\, E_I^{(R)*}(k,a) \right\rangle = \frac{1}{\sigma^2}\, \frac{16\pi}{m_p^2}\, \bar{e}^2\, e^{-\alpha\Phi^{(0)}}\, (a_0)^{\alpha\sqrt{2q}}\, \frac{2^{2(\hat{\nu}-3)}}{\pi(2-q)^{4\hat{\nu}}} \left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1+\hat{\nu})}\, (a_{R\Phi} H_{R\Phi})^{-4\hat{\nu}} \left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)} \frac{k^{2(2+\hat{\nu})}}{a^4} + \cdots, \tag{4.46}$$

$$\left\langle \epsilon_{1IJ} B_J^{(R)}(k,a) \epsilon_{1IK} B_K^{(R)*}(k,a) \right\rangle = \frac{16\pi}{m_p^2}\, \bar{e}^2\, e^{-\alpha\Phi^{(0)}}\, (a_0)^{\alpha\sqrt{2q}}\, \frac{\pi 2^{2(\hat{\nu}-1)}}{(2-q)^{4\hat{\nu}}} \left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1+\hat{\nu})}\, (a_{R\Phi} H_{R\Phi})^{-4\hat{\nu}} \left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)} \frac{k^{2(1+\hat{\nu})}}{a^2} + \cdots. \tag{4.47}$$

From eqn. (4.46), we note that $\sigma E_I^{(R)}$ remains finite in the perfect-conductivity limit. We also note that the time dependence of the large-scale expansion of eqn. (3.70) (the zero-conductivity case, for $\hat{\nu} < 1/2$) differs from that of eqn. (4.47).

We note that the expressions in (4.44) and (4.45) and in (4.46) and (4.47) satisfy

$$\left\langle \epsilon_{1IJ} B_J^{(R)}(k,a)\, \epsilon_{1IK} B_K^{(R)*}(k,a) \right\rangle = 16\pi^2\sigma^2\, \frac{a^2}{k^2} \left\langle E_I^{(R)}(k,a)\, E_I^{(R)*}(k,a) \right\rangle. \tag{4.48}$$

(This relation is a consequence of the form of the large-conductivity limit of the electric and magnetic field solutions, eqns. (4.19) and (4.20), and does not depend on the precise form of the constants of integration.)

When $\alpha = 0$ ($\hat{\nu} = 1/2$) eqns. (4.44) and (4.45) reduce to

$$\left\langle E_I^{(R)}(k,a)\, E_I^{(R)*}(k,a)\right\rangle = \frac{1}{\sigma^2}\,\frac{16\pi}{m_p^2}\,\frac{e^2}{32\pi^2}\,\frac{k^3}{a^4} + \cdots, \tag{4.49}$$

$$\left\langle \epsilon_{1IJ} B_J^{(R)}(k,a)\, \epsilon_{1IK} B_K^{(R)*}(k,a)\right\rangle = \frac{16\pi}{m_p^2}\,\frac{e^2}{2}\,\frac{k}{a^2} + \cdots. \tag{4.50}$$

From eqn. ($B20$) we note that eqn. (4.50) coincides with eqn. (3.82) (the $\sigma = 0$ case), while from eqn. ($B19$) we see that the right hand side of eqn. (4.49) is $1/(16\pi^2\sigma^2)(k/a)^2$ times the right hand side of eqn. (3.81).

From eqns. (3.83), ($B19$) and ($B20$), the energy density spectrum of the I^{th} component of the vector potential perturbation is given by

$$\epsilon_I^{(R)}(k,t) = \frac{m_p^2}{8\pi}\,\frac{1}{e^2 a^2}\left[\left\langle E_I^{(R)}(k,t)\, E_I^{(R)*}(k,t)\right\rangle + \left\langle \epsilon_{1IJ} B_J^{(R)}(k,t)\, \epsilon_{1IK} B_K^{(R)*}(k,t)\right\rangle\right], \tag{4.51}$$

where there is no implicit summation over the index I on the right hand side. We note, from eqns. (4.44) and (4.46), that the electric field part of the energy density scales as a^{-6} (unlike that of radiation). Ignoring this contribution (since it is down by σ^{-2}) we have, for $\hat{\nu} > 0$,

$$\begin{aligned}\epsilon_I^{(R)}(k,a) = {} & \left(\frac{\bar{e}}{e}\right)^2 e^{-\alpha\Phi^{(0)}}\,(a_0)^{\alpha\sqrt{2q}}\,\pi\, 2^{2\hat{\nu}-1}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \\ & \times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1-\hat{\nu})}\left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)}\frac{k^{2(1-\hat{\nu})}}{a^4} + \cdots,\end{aligned} \tag{4.52}$$

and for $\hat{\nu} < 0$,

$$\begin{aligned}\epsilon_I^{(R)}(k,a) = {} & \left(\frac{\bar{e}}{e}\right)^2 e^{-\alpha\Phi^{(0)}}\,(a_0)^{\alpha\sqrt{2q}}\,\frac{\pi\, 2^{2\hat{\nu}-1}}{(2-q)^{4\hat{\nu}}}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \\ & \times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1+\hat{\nu})}\,(a_{R\Phi}H_{R\Phi})^{-4\hat{\nu}}\left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)}\frac{k^{2(1+\hat{\nu})}}{a^4} + \cdots;\end{aligned} \tag{4.53}$$

the magnetic energy density scales like that of radiation ($\propto a^{-4}$). We note that eqn. (4.52) coincides with (3.84) (the zero-conductivity case, when $\hat{\nu} > 1/2$) while the spatial momentum dependence of eqns. (4.53) and (3.85) differ. When $\alpha = 0$ (4.52) reduces to

$$\epsilon_I^{(R)}(k,a) = \frac{k}{a^4} + \cdots, \tag{4.54}$$

which is half of the value of eqn. (3.88) (the zero-conductivity case).

5. Baryon Dominance: Perfect Conductivity

In this section we study the evolution of vector potential irregularities in the baryon dominated epoch, keeping track of the effects of the electromagnetic coupling between these perturbations and charged particles by using a simplified Ohm's law, in the large-conductivity limit. (We work with the perfect-conductivity limit equations of motion since we have not been able to solve the baryon epoch equations for the arbitrary σ case; in any case, the large σ limit is the one of interest in a first approximation, [60].) The results of this section should be compared to those of Sec. 3.5 (where we study the vanishing conductivity case).

Working in spatial Fourier space and choosing the x^1 axis to lie along the spatial momentum vector $\vec{k}$ of the perturbation of interest, we find that the longitudinal coordinate electric field and magnetic field perturbations vanish,

$$E_1^{(B)}(t) = 0, \tag{5.1}$$

$$B_1^{(B)}(t) = 0; \tag{5.2}$$

here the superscript (B) indicates that the field is the one during baryon dominance. It may be verified that (5.1) and (5.2) satisfy the $i = 1$ components of $(B28)$ and $(B29)$.

Neglecting the higher order magnetic field term in Ohm's law, the perfect-conductivity MHD approximation is that

$$\lim_{\sigma\to\infty}\left[E_i^{(B)}(t)\right] = 0, \tag{5.3}$$

$$\lim_{\sigma\to\infty}\left[\sigma E_i^{(B)}(t)\right] = \text{finite}. \tag{5.4}$$

With this approximation eqns. $(B28)$ and $(B29)$ reduce to

$$4\pi\sigma a E_i = \epsilon_{ijk}\,\partial_j B_k, \tag{5.5}$$

$$\frac{d}{dt}(aB_i) = 0. \tag{5.6}$$

The solution of the $i = I$ components of eqn. (5.6) is

$$B_I^{(B)}(t,k) = C_I^{(B)}\,a^{-1}, \tag{5.7}$$

where $C_I^{(B)}$ are k dependent constants of integration corresponding to general coordinate invariant and $U(1)$ gauge invariant solutions; the solution (5.2) satisfies the $i = 1$ component of eqn. (5.6). The $i = I$ components of eqn. (5.5) imply

$$E_I^{(B)}(t,k) = -i\,\frac{k}{4\pi\sigma a}\,\epsilon_{1IJ}\,B_J^{(B)}(t,k), \tag{5.8}$$

while the solution (5.1) satisfies the $i = 1$ component of eqn. (5.5).

If we assume that the conductivity, σ, does not jump at the radiation-baryon transition, an analysis similar to that of Sec. 3.3 or Sec. 4.3 leads to the joining conditions,

$$E_i^{(B)}(t_{BR},k) = E_i^{(R)}(t_{BR},k), \tag{5.9}$$

$$B_i^{(B)}(t_{BR},k) = B_i^{(R)}(t_{BR},k), \tag{5.10}$$

for the electric and magnetic fields at the radiation-baryon transition at $t = t_{BR}$. It is easily verified that in the large-conductivity limit, during both radiation and baryon dominance,

$$E_i(t,k) = -i\,\frac{k}{4\pi\sigma a}\,\epsilon_{1ij}\,B_j(t,k), \tag{5.11}$$

so in this limit eqn. (5.10) implies eqn. (5.9).

From (5.7) and the $i = I$ components of (5.10) we have, for the baryon epoch constant,

$$C_I^{(B)} = a_{BR}\,B_I^{(R)}(t_{BR},k), \tag{5.12}$$

where a_{BR} is the value of the scale factor at the radiation-baryon transition.

From the results of Secs. 4.4 and 4.5 we find for the large-conductivity limit of the baryon epoch transverse coordinate magnetic field two-point correlation function, when $\hat{\nu} > 0$,

$$\left\langle B_I^{(B)}(k,a)\, B_I^{(B)*}(k,a)\right\rangle = \frac{16\pi}{m_p^2}\,\bar{e}^2\, e^{-\alpha\Phi^{(0)}}\,(a_0)^{\alpha\sqrt{2q}}\,\pi\, 2^{2(\hat{\nu}-1)}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1-\hat{\nu})}\left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)}\frac{k^{2(1-\hat{\nu})}}{a^2} + \cdots, \tag{5.13}$$

and when $\hat{\nu} < 0$,

$$\left\langle B_I^{(B)}(k,a)\, B_I^{(B)*}(k,a)\right\rangle = \frac{16\pi}{m_p^2}\,\bar{e}^2\, e^{-\alpha\Phi^{(0)}}\,(a_0)^{\alpha\sqrt{2q}}\,\frac{\pi\, 2^{2(\hat{\nu}-1)}}{(2-q)^{4\hat{\nu}}}\left(\frac{q}{2-q}\right)^{-\alpha\sqrt{2q}} \times \frac{\csc^2(\hat{\nu}\pi)}{\Gamma^2(1+\hat{\nu})}\,(a_{R\Phi}H_{R\Phi})^{-4\hat{\nu}}\left(\frac{k}{\lambda}\right)^{\alpha q\sqrt{2q}/(2-q)}\frac{k^{2(1+\hat{\nu})}}{a^2} + \cdots; \tag{5.14}$$

here the ellipses denote terms that are down by positive powers of σ^{-1}. We note that the magnetic field two-point function is $\propto k^{\hat{n}}$ where $-\infty < \hat{n} < 2$ (requiring that the Fourier transform to position space be infrared finite reduces the range to $-3 < \hat{n} < 2$). The energy density carried by these perturbations has a similar spatial momentum dependence.

The form of eqns. (5.13) and (5.14) is identical to that of the corresponding radiation epoch expressions, (4.45) and (4.47). The transverse coordinate electric field two-point correlation function may be derived from these equations by using (5.8); they are identical in form to the corresponding radiation epoch expressions, (4.44) and (4.46). The energy density carried by the magnetic field during baryon dominance is given by equations identical in form to the corresponding radiation epoch expressions, (4.52) and (4.53).

6. Predictions and Observational Constraints

Ignoring the Maxwell-scalar field term in the action, the exponential potential scalar field inflation model we consider has two free parameters: the redshift of reheating $z_{R\Phi}$; and the late-time energy density power spectrum index n [41]. (We have assumed that the universe is spatially flat, with a given Hubble parameter and the standard value for the redshift of the radiation-baryon transition.) In Ref. [41] we have determined $z_{R\Phi}$ as a function of n by requiring that the theoretical expression for the mean square departure velocity from Hubble expansion agree with the observational data. The inflation epoch Maxwell-scalar field coupling, eqns. (2.1) and (2.21), brings into play another free parameter, α. Given $z_{R\Phi}$ as a function of n, we need to specify both n and α to determine the present magnetic field.

6.1 Technical Preliminaries and the Scale-Invariant Magnetic Field

To compare to the observational upper bound on the intergalactic magnetic field we need to determine the position space magnetic field. Since the magnetic field perturbation is general coordinate invariant we may, in a first approximation, evaluate this expression in a constant time synchronous gauge hypersurface. This is simpler than what must be done to compare the fluid perturbations to the observations, Sec. V of Ref. [41].

We define the baryon epoch, position space, transverse proper magnetic field, on a coordinate length scale R, about a point $\vec{x}_0$, at time t_N, by

$$[a(t_N)]^{-1}\, B_I^{(B)}(t_N|R) = \int_{-\infty}^{\infty} [a(t_N)]^{-1}\, B_I^{(B)}(t_N, \vec{x})\, W(|\vec{x}_0 - \vec{x}|, R)\, d^3x, \qquad (6.1)$$

where the normalized Gaussian window function,

$$W(|\vec{x}_0 - \vec{x}|, R) = (2\pi R^2)^{-3/2} \exp\left[-|\vec{x}_0 - \vec{x}|^2/(2R^2)\right], \qquad (6.2)$$

is used to suppress the contribution of large spatial momentum modes. Fourier expanding the magnetic field and integrating, we find,

$$\left\langle \left[[a(t_N)]^{-1} B_I^{(B)}(t_N|R) \right]^2 \right\rangle = 4 \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \, e^{-k^2 R^2} \, [a(t_N)]^{-2} \left\langle B_I^{(B)}(t_N, k) \, B_I^{(B)*}(t_N, k) \right\rangle , \tag{6.3}$$

where the factor of 4 on the right hand side is a consequence of our Fourier expansion convention and there is no implicit summation over the index I in this equation (and in similar equations that follow).

Defining

$$r = \sqrt{(1-n)(3-n)}, \tag{6.4}$$

where

$$n = \frac{2-3q}{2-q}, \tag{6.5}$$

is the baryon epoch, large-scale fractional energy irregularity density power spectrum index [$\langle \delta_B(k) \delta_B^*(k) \rangle \propto k^n$, eqn. (5.43) of Ref. [41]], we find, from eqn. (5.13), for $\hat{\nu} > 0$,

$$\begin{aligned} \left\langle \left[[a(t_N)]^{-1} B_I^{(B)}(t_N|\mathcal{R}) \right]^2 \right\rangle =& \frac{8e^2}{m_p^2 \pi^2} \left(\frac{4}{3-n} \right)^{\alpha r} \Gamma\!\left(2 - \frac{\alpha r}{2}\right) \Gamma^2\!\left(\frac{1}{2} + \frac{\alpha r}{2}\right) \\ & \times (1 + z_{R\Phi})^{\alpha r} (H_{BR})^4 (1 + z_{BR})^{-(6+\alpha r/2)} \\ & \times (1 + z_N)^{(6-\alpha r/2)} (\mathcal{R} H_N)^{-(4-\alpha r)}, \end{aligned} \tag{6.6}$$

which is valid for the range $-1 < \alpha r < 4$, where the second inequality follows from the requirement that the integral in (6.3) not diverge in the infrared (we have taken αr to be real) and the first is a rewriting of $\hat{\nu} > 0$. In this equation $z_{R\Phi}$, z_{BR}, and z_N are the redshifts of the reheating transition, the radiation-baryon transition, and the time t_N; H_{BR} and H_N are the values of the Hubble parameter at the radiation-baryon transition and at the time t_N; and $\mathcal{R} = a(t_N)R$ is the proper length scale. When $\hat{\nu} < 0$ we find, from (5.14),

$$\begin{aligned} \left\langle \left[[a(t_N)]^{-1} B_I^{(B)}(t_N|\mathcal{R}) \right]^2 \right\rangle =& \frac{8e^2}{m_p^2 \pi^2} \left(\frac{4}{3-n} \right)^{-(2+\alpha r)} \Gamma\!\left(3 + \frac{\alpha r}{2}\right) \Gamma^2\!\left(-\frac{1}{2} - \frac{\alpha r}{2}\right) \\ & \times (1 + z_{R\Phi})^{-(2+\alpha r)} (H_{BR})^4 (1 + z_{BR})^{-(5-\alpha r/2)} \end{aligned}$$

$$\times\,(1+z_N)^{(7+\alpha r/2)}(\mathcal{R}H_N)^{-(6+\alpha r)}, \tag{6.7}$$

which is valid for $-6 < \alpha r < -1$ (the lower limit is a consequence of requiring infrared finiteness). In deriving eqns. (6.6) and (6.7) we have used

$$\bar{e}^2\, e^{-\alpha\Phi^{(0)}}\,(a_0)^{\alpha\sqrt{2q}} = e^2\,(a_{R\Phi})^{\alpha\sqrt{2q}}, \tag{6.8}$$

which follows from eqns. (2.21) – (2.23) and (3.45). It is interesting to note that replacing αr by $-(2+\alpha r)$ in eqn. (6.6) converts this equation to eqn. (6.7); the physical reason for this symmetry is not yet obvious.

The spatial length scale dependence of the magnetic part of the energy density is determined by eqn. (6.6) or (6.7). Following the nomenclature used for energy density perturbations [74] we might call the case when $\epsilon_I^{(B)}(\mathcal{R}) \propto (\mathcal{R})^{-2}$ the scale-invariant case. This results when $\alpha = 2/r$ [eqn. (6.6)] or $= -4/r$ [eqn. (6.7)]. We emphasize that, within the allowed range of αr, the magnetic field can have more power on large scales or on small scales than in the scale-invariant case; the magnetic two-point function is $\propto \mathcal{R}^{-(\hat{n}+3)}$ where $0 < \hat{n}+3 < 5$ (the 'free' $\alpha = 0$ case corresponds to $\hat{n} = 1$ while $\hat{n} = -1$ is the scale-invariant case).

6.2 Magnitude of the Present Magnetic Field

In the model under consideration the universe is spatially flat so $\Omega_{\rm now} = 1$. We shall also make use of the hot big bang model values,

$$H_{BR} = 1.4\times 10^{-54}\, h^4\, m_p, \tag{6.9}$$

$$z_{BR} = 4\times 10^4\, h^2, \tag{6.10}$$

where we have taken $H_{\rm now} = 100\, h\, {\rm km\, s^{-1} Mpc^{-1}}$. To ensure that the electromagnetic action is conventionally normalized we set

$$e^2 = \frac{1}{4}\, m_p^2; \tag{6.11}$$

with this choice, and using the results of Appendix B, the time-time component of the

Maxwell stress-energy tensor [eqn. (3.7)] for the transverse part of the field, during baryon dominance, is also conventionally normalized,

$$T_{(M)00}(t,\vec{x}) = \frac{\delta_{IJ}}{8\pi}\left[\frac{E_I(t,\vec{x})}{a(t)}\,\frac{E_J(t,\vec{x})}{a(t)} + \frac{B_I(t,\vec{x})}{a(t)}\,\frac{B_J(t,\vec{x})}{a(t)}\right] \tag{6.12}$$

(here E_I/a and B_I/a are the proper transverse electric and magnetic fields).

Taking t_N to be the present time $t_{\rm now}$, focussing on a representative proper length scale $\mathcal{R} = 10^{-3}/H_{\rm now}$, and using $z_{R\Phi} \gg 1$, we find, from eqn. (6.6), for $-1 < \alpha r < 4$,

$$\left\langle \left[[a(t_{\rm now})]^{-1} B_I^{(B)}(t_{\rm now}|\mathcal{R} = 10^{-3}/H_{\rm now})\right]^2 \right\rangle^{1/2} \simeq 3\times 10^{-59}\Gamma^{1/2}\!\left(2 - \frac{\alpha r}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\alpha r}{2}\right) \times \left[\frac{5\times 10^4(3-n)h}{z_{R\Phi}}\right]^{-\alpha r/2} h^2\,{\rm G}, \tag{6.13}$$

and from eqn. (6.7), for $-6 < \alpha r < -1$,

$$\left\langle \left[[a(t_{\rm now})]^{-1} B_I^{(B)}(t_{\rm now}|\mathcal{R} = 10^{-3}/H_{\rm now})\right]^2 \right\rangle^{1/2} \simeq 3\times 10^{-59}\Gamma^{1/2}\!\left(3 + \frac{\alpha r}{2}\right)\Gamma\left(-\frac{1}{2} - \frac{\alpha r}{2}\right) \times \left[\frac{5\times 10^4(3-n)h}{z_{R\Phi}}\right]^{1+\alpha r/2} h^2\,{\rm G}. \tag{6.14}$$

The value of the numerical prefactor in these equations is a consequence of $H_{BR}^2/[(1+z_{BR})^3(\mathcal{R}H_{\rm now})^2] \simeq 3.1\times 10^{-116} h^2 m_p^2 \simeq 6.6\times 10^{-59} h^2{\rm G}$. (We note that these expressions are for one transverse component of the field; the total RMS field is a factor of $\sqrt{2}$ larger.)

One might expect the expressions of eqns. (6.6) and (6.7) to hold on the small spatial scale $\mathcal{R} = 10^{-3}/H_{\rm now}$ for sufficiently large baryon epoch conductivity σ. We note that the radiation epoch expressions, eqns. (4.45) and (4.47), assume that f, eqn. (4.14), is small; for large enough radiation epoch conductivity σ [$\sigma \gg (k/a)^2/(8\pi H)$] these expressions are valid on small scales [provided $k/a \ll \sqrt{8\pi\sigma H}$]. It might not be unreasonable to expect that a similar argument holds in the baryon epoch. However, in the absence of a baryon epoch solution for arbitrary σ (the analysis of Sec. 5 assumes that σ is large) and in the absence of a good observational estimate of the value of the conductivity, this argument cannot be reliably quantified. We also emphasize that although a perturbation of present scale $\mathcal{R} = 10^{-3}/H_{\rm now}$ reentered the Hubble scale in the radiation epoch, unlike for energy density perturbations [80], the momentum dependence of the primeval magnetic field spectrum, eqns. (5.13) and (5.14), also holds on such small scales. (The neglect of Ohmic dissipation during baryon dominance means that these expressions do not hold on very small scales.)

This is probably the appropriate point to discuss how this model fits in with the low-energy electroweak model. After the electroweak transition, the electromagnetic photon is a linear combination of the abelian weak hypercharge gauge field and the nonabelian weak isospin gauge field. The present electromagnetic magnetic field two-point correlation function is a function of the magnetic field correlation functions of the two fields whose linear combination makes up the photon, as well as of the weak mixing angle θ_W. To try to get some idea of the magnitude of the present electromagnetic magnetic field we consider an idealized model: we assume that above the phase transition the Higgs field is small (~ 0), and that the transition happens rapidly compared to the Hubble timescale at this epoch. Assuming that the gauge fields are also small above the transition (and assuming that the effects of the fermions may be summarized by an effective conductivity) we may, in a first approximation, work with the quadratic part of the standard electroweak Lagrangian density. The hypercharge gauge field part of this Lagrangian density is identical to the gauge field part of the Lagrangian density we have used in the radiation and baryon epochs (provided we replace the coupling constant we use, e, by the coupling constant $e/\cos\theta_W$), while the isospin gauge field part of the quadratic electroweak Lagrangian density reproduces the Lagrangian density we use in triplicate (provided we replace e by $e/\sin\theta_W$). These redefinitions of the coupling constant mean that the pretransition hypercharge and isospin magnetic field two-point functions will be a factor of $1/\cos^2\theta_W$ and a factor of $1/\sin^2\theta_W$ times the expression we have found [note the coupling constant dependence of eqns. (6.6) and (6.7)]. Now after the transition the electromagnetic magnetic field two-point function is a linear combination of the hypercharge and isospin magnetic field two-point functions (i.e., the two fields whose linear combination makes up the electromagnetic photon) with coefficients $\cos^2\theta_W$ and $\sin^2\theta_W$. These factors will compensate for the factors that come from the coupling constant redefinitions so just after the completion of the transition the electromagnetic magnetic field two-point function will be a factor of 2 larger than we have estimated. (This argument assumes that the various magnetic field correlation functions are unaffected by the transition.) To get a lower limit on the electromagnetic magnetic field one might assume that only the hypercharge magnetic field contributes; then our expression for the electromagnetic magnetic field remains unchanged. It would be useful to see if one of these estimates can be justified by a real calculation.

In Ref. [41] we have estimated the redshift of reheating $z_{R\Phi}$, as a function of the power spectrum index n, by comparing the theoretical expression for the mean square departure velocity from homogeneous Hubble expansion to the observational data. Some order-of-magnitude numerical values of interest are (eqns. (5.77) and (5.78) of Ref. [41]),

$$z_{R\Phi} \simeq \begin{cases} 5.4 \times 10^{29} \sqrt{\epsilon}\, h^{\epsilon^2/2} & (n = 1 - 2\epsilon^2) \\ 3.8 \times 10^{26}\ h^{1/9} & (n = 1/2) \\ 2.2 \times 10^{24}\ h^{1/5} & (n = 0) \\ 2.9 \times 10^{22}\ h^{3/11} & (n = -1/2) \\ 1.2 \times 10^{21}\ h^{1/3} & (n = -1), \end{cases} \tag{6.15}$$

where ϵ is a small positive parameter used to 'regulate' the de Sitter inflation limit divergence of the large time energy density power spectrum [41] (and should not be confused with the dielectric 'constant' ϵ).

Denoting $\left\langle [\{a(t_{\rm now})\}^{-1} B_I^{(B)}(t_{\rm now}|\mathcal{R} = 10^{-3}/H_{\rm now})]^2 \right\rangle$ by Θ^2, we have, for $\alpha r = 4 - 2\bar{\epsilon}$ or $\alpha r = -6 + 2\bar{\epsilon}$ (where $\bar{\epsilon}$ is a small positive parameter),

$$
\Theta \simeq \begin{cases} 1.2\times 10^{-9}\epsilon\,\bar\epsilon^{-1/2}\,\mathrm{G} & (n=1-2\epsilon^2;\alpha=2\epsilon^{-1}\ \text{or}\ -3\epsilon^{-1}) \\ 3.7\times 10^{-16}\,\bar\epsilon^{-1/2}\,h^{2/9}\,\mathrm{G} & (n=1/2;\alpha\simeq 3.6\ \text{or}\ -5.4) \\ 8.5\times 10^{-21}\,\bar\epsilon^{-1/2}\,h^{2/5}\,\mathrm{G} & (n=0;\alpha\simeq 2.3\ \text{or}\ -3.5) \\ 1.1\times 10^{-24}\,\bar\epsilon^{-1/2}\,h^{6/11}\,\mathrm{G} & (n=-1/2;\alpha\simeq 1.7\ \text{or}\ -2.6) \\ 1.4\times 10^{-27}\,\bar\epsilon^{-1/2}\,h^{2/3}\,\mathrm{G} & (n=-1;\alpha\simeq 1.4\ \text{or}\ -2.1), \end{cases} \tag{6.16}
$$

where the first value of α is from $\alpha r = 4-2\bar\epsilon$ and the second from $\alpha r = -6+2\bar\epsilon$. We have suppressed a factor of $[5\times 10^4(3-n)h/z_{R\Phi}]^{\bar\epsilon}$ in these expressions; this is only slightly smaller than unity for small enough $\bar\epsilon$ (say $\sim 10^{-2}$). [We also note, from eqns. (6.6) and (6.7), that for these values of αr the transverse proper RMS magnetic field Θ is almost insensitive to $\mathcal{R}$.] For $\alpha r = 7/2$ or $\alpha r = -11/2$,

$$
\Theta \simeq \begin{cases} 1.2\times 10^{-15}\epsilon^{7/8}\,h^{1/4}\,\mathrm{G} & (n=1-2\epsilon^2;\alpha\simeq 1.8\epsilon^{-1}\ \text{or}\ -2.8\epsilon^{-1}) \\ 2.5\times 10^{-21}\ h^{4/9}\,\mathrm{G} & (n=1/2;\alpha\simeq 3.1\ \text{or}\ -4.9) \\ 2.2\times 10^{-25}\ h^{3/5}\,\mathrm{G} & (n=0;\alpha\simeq 2.0\ \text{or}\ -3.2) \\ 8.7\times 10^{-29}\ h^{8/11}\,\mathrm{G} & (n=-1/2;\alpha\simeq 1.5\ \text{or}\ -2.4) \\ 2.6\times 10^{-31}\ h^{5/6}\,\mathrm{G} & (n=-1;\alpha\simeq 1.2\ \text{or}\ -1.9), \end{cases} \tag{6.17}
$$

for $\alpha r = 5/2$ or $\alpha r = -9/2$,

$$
\Theta \simeq \begin{cases} 2.5\times 10^{-28}\epsilon^{5/8}\,h^{3/4}\,\mathrm{G} & (n=1-2\epsilon^2;\alpha\simeq 1.3\epsilon^{-1}\ \text{or}\ -2.3\epsilon^{-1}) \\ 2.2\times 10^{-32}\ h^{8/9}\,\mathrm{G} & (n=1/2;\alpha\simeq 2.2\ \text{or}\ -4.0) \\ 2.7\times 10^{-35}\ h\,\mathrm{G} & (n=0;\alpha\simeq 1.4\ \text{or}\ -2.6) \\ 1.0\times 10^{-37}\ h^{12/11}\,\mathrm{G} & (n=-1/2;\alpha\simeq 1.1\ \text{or}\ -2.0) \\ 1.6\times 10^{-39}\ h^{7/6}\,\mathrm{G} & (n=-1;\alpha\simeq 0.89\ \text{or}\ -1.6), \end{cases} \tag{6.18}
$$

for the scale-invariant values, $\alpha r = 2$ or $\alpha r = -4$,

$$
\Theta \simeq \begin{cases} 1.4\times 10^{-34}\epsilon^{1/2}\ h\,\mathrm{G} & (n=1-2\epsilon^2;\alpha=\epsilon^{-1}\ \text{or}\ -2\epsilon^{-1}) \\ 8.0\times 10^{-38}\ h^{10/9}\,\mathrm{G} & (n=1/2;\alpha\simeq 1.8\ \text{or}\ -3.6) \\ 3.9\times 10^{-40}\ h^{6/5}\,\mathrm{G} & (n=0;\alpha\simeq 1.2\ \text{or}\ -2.3) \\ 4.4\times 10^{-42}\ h^{14/11}\,\mathrm{G} & (n=-1/2;\alpha\simeq 0.88\ \text{or}\ -1.7) \\ 1.6\times 10^{-43}\ h^{4/3}\,\mathrm{G} & (n=-1;\alpha\simeq 0.7\ \text{or}\ -1.4), \end{cases} \tag{6.19}
$$

for $\alpha r = 3/2$ or $\alpha r = -7/2$,

$$\Theta \simeq \begin{cases} 9.1 \times 10^{-41} \epsilon^{3/8}\, h^{5/4}\,\mathrm{G} & (n = 1 - 2\epsilon^2; \alpha \simeq 0.75\epsilon^{-1} \text{ or } -1.8\epsilon^{-1}) \\ 3.3 \times 10^{-43}\; h^{4/3}\,\mathrm{G} & (n = 1/2; \alpha \simeq 1.3 \text{ or } -3.1) \\ 6.1 \times 10^{-45}\; h^{7/5}\,\mathrm{G} & (n = 0; \alpha \simeq 0.87 \text{ or } -2.0) \\ 2.1 \times 10^{-46}\; h^{16/11}\,\mathrm{G} & (n = -1/2; \alpha \simeq 0.65 \text{ or } -1.5) \\ 1.7 \times 10^{-47}\; h^{3/2}\,\mathrm{G} & (n = -1; \alpha \simeq 0.53 \text{ or } -1.2), \end{cases} \tag{6.20}$$

for $\alpha r = 1/2$ or $\alpha r = -5/2$,

$$\Theta \simeq \begin{cases} 5.3 \times 10^{-53} \epsilon^{1/8}\, h^{7/4}\,\mathrm{G} & (n = 1 - 2\epsilon^2; \alpha \simeq 0.25\epsilon^{-1} \text{ or } -1.3\epsilon^{-1}) \\ 8.2 \times 10^{-54}\; h^{16/9}\,\mathrm{G} & (n = 1/2; \alpha \simeq 0.45 \text{ or } -2.2) \\ 2.2 \times 10^{-54}\; h^{9/5}\,\mathrm{G} & (n = 0; \alpha \simeq 0.29 \text{ or } -1.4) \\ 7.0 \times 10^{-55}\; h^{20/11}\,\mathrm{G} & (n = -1/2; \alpha \simeq 0.22 \text{ or } -1.1) \\ 3.1 \times 10^{-55}\; h^{11/6}\,\mathrm{G} & (n = -1; \alpha \simeq 0.18 \text{ or } -0.88), \end{cases} \tag{6.21}$$

for the 'free' case when $\alpha = 0$ (or when $\alpha r = -2$),

$$\Theta \simeq 5.3 \times 10^{-59}\, h^2\,\mathrm{G}, \tag{6.22}$$

and when $\alpha r = -1/2$ or $\alpha r = -3/2$,

$$\Theta \simeq \begin{cases} 7.5 \times 10^{-65} \epsilon^{-1/8}\, h^{9/4}\,\mathrm{G} & (n = 1 - 2\epsilon^2; \alpha = -0.25\epsilon^{-1} \text{ or } -0.75\epsilon^{-1}) \\ 4.9 \times 10^{-64}\; h^{20/9}\,\mathrm{G} & (n = 1/2; \alpha \simeq -0.45 \text{ or } -1.3) \\ 1.9 \times 10^{-63}\; h^{11/5}\,\mathrm{G} & (n = 0; \alpha \simeq -0.29 \text{ or } -0.87) \\ 5.7 \times 10^{-63}\; h^{24/11}\,\mathrm{G} & (n = -1/2; \alpha \simeq -0.22 \text{ or } -0.65) \\ 1.3 \times 10^{-62}\; h^{13/6}\,\mathrm{G} & (n = -1; \alpha \simeq -0.18 \text{ or } -0.53). \end{cases} \tag{6.23}$$

We emphasize that these numbers are only order-of-magnitude estimates.

It is interesting to note that in the de Sitter inflation limit (when $n \to 1$ or $\epsilon \to 0$) the late time magnetic field might decrease [eqns. (6.16) – (6.21)], remain unchanged [eqn. (6.22)], or increase [eqn. (6.23)]; this should be contrasted with the behaviour of the late-time energy density perturbation [which increases, because the time derivative of the homogeneous part of the inflation epoch scalar field decreases] and the late time peculiar velocity and gravitational wave perturbations [which remain unchanged] in this limit [41,42,66,64].

Ignoring the $z_{R\Phi}$ dependent factor in eqns. (6.13) and (6.14) we have $\Theta \sim 10^{-59}$G [which is what is found for the 'free' ($\alpha = 0$) case, eqn. (6.22)]. For the $\alpha \neq 0$ case this value is modified by a factor of order $(10^5/z_{R\Phi})^{\hat{r}}$, $-2 < \hat{r} < 1/2$; $10^5/z_{R\Phi} \ll 1$ (we emphasize that $z_{R\Phi}$ is fixed by comparing the velocity perturbation to observational data [41]) so when $\hat{r}$ is negative $\Theta > 10^{-59}$G. This is the reason for the 'large' (relative to 10^{-59}G) value of Θ in parts of parameter space. The first entry in eqn. (6.16) is the largest value tabulated both because $\hat{r}(= -2 + \bar{\epsilon})$ is most negative (for sufficiently small $\bar{\epsilon}$) and because $1/z_{R\Phi}$ is smallest [from eqn. (6.15), provided ϵ is not too small] in this region of parameter space.

We note, from eqns. (6.6) and (6.7), that for a given, allowed, value of αr, the field increases as $\mathcal{R}$ is decreased; on small enough scales linear theory becomes invalid and interactions (including Ohmic dissipation) which we have ignored are important. We note that when $\alpha r = 4 - 2\bar{\epsilon}$ or $= -6 + 2\bar{\epsilon}$ the field is almost $\mathcal{R}$ independent, and that as we progress from eqn. (6.16) to eqn. (6.23) the $\mathcal{R}$ dependence of the field becomes more pronounced with the $\mathcal{R}$ dependent factor increasing as $\mathcal{R}$ is decreased. Since the $\mathcal{R}$ independent prefactor in the expression for the field decreases along this progression there does not seem to be a serious problem on small scales, e.g., for $\mathcal{R} = 10^{-5}/H_{\rm now}$ the numbers in eqns. (6.23) should be increased by a factor of $10^{5.5}$ and for $\mathcal{R} = 10^{-7}/H_{\rm now}$ by a factor of 10^{11}. We also note that since $1\,{\rm G}^2 \simeq 1.1 \times 10^{-21}\,{\rm g\,cm}^{-3}$, the present energy density in the magnetic field is very small, even compared to that in the microwave background, $\rho_{R,{\rm now}} \simeq 4.5 \times 10^{-34}\,{\rm g\,cm}^{-3}$. We shall see that when αr is sufficiently close to $-6 + 2\bar{\epsilon}$ and n is sufficiently close to $1 - 2\epsilon^2$ (but not unity), the inflation epoch large-scale vector potential energy density exceeds that in the homogeneous scalar field and so this region of parameter space must be excluded. (From the discussion in Sec. 2.4 we find that in this case the inflation epoch energy density scales like a^2 while for $n = 1 - 2\epsilon^2$ and $\alpha r = 4 - 2\bar{\epsilon}$ it scales like a^0.)

An intergalactic field of present strength $\sim 10^{-10}$G, that has been amplified by isotropic collapse of the galaxy, would lead to a galactic field $\sim 10^{-6}$G, roughly what is now observed (this estimate assumes that the amplification of the field by the differential rotation of the galaxy is negligible; for the range of the theoretically desired intergalactic field see Sec. 17 of Ref. [62], Sec. 12.1 of Ref. [31] and Sec. 15.II of Ref. [3]). If the amplification by differential rotation of the galaxy after (anisotropic) collapse is accounted for one needs a smaller intergalactic field, $\sim 10^{-13}$ – 10^{-12}G, [34,35]. There does not seem to be a definite estimate of the present value of the intergalactic field that would provide a seed for a dynamo to possibly amplify to a galactic field $\sim 10^{-6}$G. What is usually estimated is the (somewhat model-dependent) needed seed field in a protogalaxy after collapse (in particular, the needed seed field would depend on how the dynamo reaches steady-state); if this seed were the consequence of a primeval field then the required value of this primeval field would presumably depend on the epoch at which collapse occurs (which introduces further model dependence). An intergalactic field of present strength $\sim 10^{-22}$ – 10^{-16}G would seem to cover most of the range of magnitude of the needed seed for a galactic dynamo (for the range of values see Sec. 13.III of Ref. [3] and Refs. [38,50]).

For most of the allowed range of parameter space the intergalactic field we have found, eqns. (6.16) – (6.23), is well below the theoretically desired values; however, for αr near 4 and n near 1 we do get a large enough intergalactic field for amplification by galactic collapse to produce the observed galactic field [including the factor mentioned below eqn. (6.16) and taking $\epsilon = 0.1$ and $\bar{\epsilon} = 0.01$, the first entry in eqn. (6.16) becomes $\Theta \simeq 6.5 \times 10^{-10}$G; we have noted above that the total field is a factor of $\sqrt{2}$ larger and that the electroweak transition might increase this by another factor of $\sqrt{2}$ — it is probably unlikely that a more realistic version of the model considered here will lead to a maximum field much larger than 10^{-9}G], and it is somewhat easier to get an intergalactic magnetic field larger than or of the order of that theoretically desired for the galactic dynamo process. [While it might seem somewhat disturbing that for our preferred region in parameter space $\alpha = 2\epsilon^{-1}$, eqn. (6.16), and so the Maxwell-scalar coupling of eqn. (2.21) would seem to depend on a parameter of the potential, we should note that ϵ is really a given number, say 0.1, and by $\alpha = 2\epsilon^{-1}$ we really mean a fixed number, $\alpha = 20$, which is no better or worse than choosing $\alpha \simeq 3.6$ or $\alpha \simeq 2.3$ or any other value.] In this case the inflation epoch Maxwell model is strongly coupled at early times. In the weakly coupled case the maximum Θ (consistent with the inflation epoch Maxwell field energy density remaining below that in the homogeneous scalar field) is $\sim 10^{-23}$G, for $\alpha \sim -2.6$ and $n = -1/2$ [eqn. (6.16)].

6.3 Consistency

During radiation and baryon dominance the magnetic field energy density redshifts like that of radiation $\propto a^{-4}$, eqns. (4.52) and (4.53) and eqns. (5.13) and (5.14) [combined with eqn. (4.51)]. At the present time the field energy density is only a fraction of the energy density in the microwave background radiation and so in the baryon and radiation epochs the magnetic field energy density is always subdominant. At redshifts greater than that of reheating (which is when 'radiation' is produced), i.e., during inflation, one must check whether the assumption used, that the large-scale Maxwell field energy density is significantly smaller than the homogeneous scalar field energy density (responsible for powering the inflation expansion), is valid. One might also wish to check whether the linear perturbation approximation (for the large-scale Maxwell field) holds during inflation.

The perturbation approximation for the Maxwell field may be checked by estimating

$$\chi^2 = \frac{16\pi}{m_p^2}\frac{k^3}{\pi^2}\langle A_I(t,k)\, A_I^*(t,k)\rangle\, (a_{\rm now})^{-2} \tag{6.24}$$

(there is no implicit summation over the index I on the right hand side of this equation). χ is a measure of the position space magnitude of the vector potential; the factor k^3/π^2 roughly translates the momentum space correlation function to a position space one, the factor $16\pi/m_p^2$ follows from the fact that the spatial irregularity expansion is an expansion in the coupling constant $\sqrt{16\pi/m_p^2}$, and if one wishes one may set $a_{\rm now} = 1$. (If χ were large it would be incorrect to work in a perturbation expansion about a spatially isotropic background.) During inflation, on large scales, we find from eqns. (2.2) and (2.63), for $\hat{\nu} > 0$ ($\alpha r > -1$),

$$\chi \simeq 2.8 \times 10^{-63} \Gamma\left(\frac{1}{2} + \frac{\alpha r}{2}\right) \left(\frac{4}{3-n}\right)^{\alpha r/2} \left(\frac{k}{aH}\right)^{1-\alpha r/2} \times (1+z)^{-(2-\alpha r)/(3-n)} (1+z_{R\Phi})^{(5-n-\alpha r)/(3-n)}, \tag{6.25}$$

and from eqns. (2.2) and (2.65), for $\hat{\nu} < 0$ ($\alpha r < -1$),

$$\chi \simeq 2.8 \times 10^{-63} \Gamma\left(-\frac{1}{2} - \frac{\alpha r}{2}\right) \left(\frac{4}{3-n}\right)^{-(1+\alpha r/2)} \left(\frac{k}{aH}\right)^{2+\alpha r/2} \times (1+z)^{-(2-\alpha r)/(3-n)} (1+z_{R\Phi})^{(5-n-\alpha r)/(3-n)}; \tag{6.26}$$

the symbols in these equations have been defined in the previous section and the small numerical prefactor is a consequence of $H_{BR}/[m_p(1+z_{BR})^2] \simeq 8.8 \times 10^{-64}$.

A perturbation with spatial momentum k, where

$$\frac{k}{a_{R\Phi} H_{R\Phi}} = \frac{4\pi \times 10^5 h}{(1+z_{R\Phi})}, \tag{6.27}$$

just before reheating (at $z = z_{R\Phi}$), will evolve to a present proper length scale

$$2\pi \frac{a_{\rm now}}{k} = \frac{10^{-3}}{H_{\rm now}}, \tag{6.28}$$

which is the scale on which we have evaluated the present magnetic field in Sec. 6.2. Setting $z = z_{R\Phi}$ in eqns. (6.25) and (6.26) and using eqn. (6.27), we find, for $\alpha r > -1$,

$$\chi \simeq 3.5 \times 10^{-57} \Gamma\left(\frac{1}{2} + \frac{\alpha r}{2}\right) \left(\frac{4}{3-n} \frac{a_{R\Phi} H_{R\Phi}}{k}\right)^{\alpha r/2} h, \tag{6.29}$$

and for $\alpha r < -1$,

$$\chi \simeq 3.5 \times 10^{-57} \Gamma\left(-\frac{1}{2} - \frac{\alpha r}{2}\right) \left(\frac{4}{3-n} \frac{a_{R\Phi} H_{R\Phi}}{k}\right)^{-(1+\alpha r/2)} h. \tag{6.30}$$

We note that replacing αr by $-(2+\alpha r)$ in eqn. (6.29) converts it to eqn. (6.30).

Evaluating eqns. (6.29) and (6.30), we find, for $\alpha r = 4 - 2\bar{\epsilon}$ or $\alpha r = -6 + 2\bar{\epsilon}$,

$$\chi \simeq \begin{cases} 3.4 \times 10^{-9} \epsilon\, h^{-1} & (n = 1 - 2\epsilon^2) \\ 1.1 \times 10^{-15}\, h^{-7/9} & (n = 1/2) \\ 2.5 \times 10^{-20}\, h^{-3/5} & (n = 0) \\ 3.2 \times 10^{-24}\, h^{-5/11} & (n = -1/2) \\ 4.3 \times 10^{-27}\, h^{-1/3} & (n = -1) \end{cases} \tag{6.31}$$

[where the corresponding values of α are listed in eqn. (6.16) and the values of α for the following equations may be found in eqns. (6.17) – (6.23)]. We note that in eqn. (6.31) we have suppressed a factor of $[\{4/(3-n)\}\{a_{R\Phi} H_{R\Phi}/k\}]^{-\bar{\epsilon}}$, which is only slightly smaller than unity for small enough $\bar{\epsilon}$ ($\sim 10^{-2}$). For $\alpha r = 7/2$ or $\alpha r = -11/2$,

$$\chi \simeq \begin{cases} 3.0 \times 10^{-15} \epsilon^{7/8}\, h^{-3/4} & (n = 1 - 2\epsilon^2) \\ 6.3 \times 10^{-21}\, h^{-5/9} & (n = 1/2) \\ 5.5 \times 10^{-25}\, h^{-2/5} & (n = 0) \\ 2.2 \times 10^{-28}\, h^{-3/11} & (n = -1/2) \\ 6.5 \times 10^{-31}\, h^{-1/6} & (n = -1), \end{cases} \tag{6.32}$$

for $\alpha r = 5/2$ or $\alpha r = -9/2$,

$$\chi \simeq \begin{cases} 2.7 \times 10^{-27} \epsilon^{5/8}\, h^{-1/4} & (n = 1 - 2\epsilon^2) \\ 2.3 \times 10^{-31}\, h^{-1/9} & (n = 1/2) \\ 2.9 \times 10^{-34} & (n = 0) \\ 1.1 \times 10^{-36}\, h^{1/11} & (n = -1/2) \\ 1.7 \times 10^{-38}\, h^{1/6} & (n = -1), \end{cases} \tag{6.33}$$

for the scale-invariant magnetic field values, $\alpha r = 2$ or $\alpha r = -4$,

$$\chi \simeq \begin{cases} 2.7 \times 10^{-33} \epsilon^{1/2} & (n = 1 - 2\epsilon^2) \\ 1.5 \times 10^{-36}\, h^{1/9} & (n = 1/2) \\ 7.3 \times 10^{-39}\, h^{1/5} & (n = 0) \\ 8.2 \times 10^{-41}\, h^{3/11} & (n = -1/2) \\ 3.0 \times 10^{-42}\, h^{1/3} & (n = -1), \end{cases} \tag{6.34}$$

for $\alpha r = 3/2$ or $\alpha r = -7/2$,

$$\chi \simeq \begin{cases} 2.8 \times 10^{-39} \epsilon^{3/8}\, h^{1/4} & (n = 1 - 2\epsilon^2) \\ 1.0 \times 10^{-41}\ h^{1/3} & (n = 1/2) \\ 1.9 \times 10^{-43}\ h^{2/5} & (n = 0) \\ 6.6 \times 10^{-45}\ h^{5/11} & (n = -1/2) \\ 5.5 \times 10^{-46}\ h^{1/2} & (n = -1), \end{cases} \tag{6.35}$$

for $\alpha r = 1/2$ or $\alpha r = -5/2$,

$$\chi \simeq \begin{cases} 4.1 \times 10^{-51} \epsilon^{1/8}\, h^{3/4} & (n = 1 - 2\epsilon^2) \\ 6.4 \times 10^{-52}\ h^{7/9} & (n = 1/2) \\ 1.7 \times 10^{-52}\ h^{4/5} & (n = 0) \\ 5.5 \times 10^{-53}\ h^{9/11} & (n = -1/2) \\ 2.4 \times 10^{-53}\ h^{5/6} & (n = -1), \end{cases} \tag{6.36}$$

for the 'free' case when $\alpha = 0$ (or when $\alpha r = -2$),

$$\chi \simeq 6.2 \times 10^{-57}\ h, \tag{6.37}$$

and when $\alpha r = -1/2$ or $\alpha r = -3/2$,

$$\chi \simeq \begin{cases} 1.3 \times 10^{-62} \epsilon^{-1/8}\, h^{5/4} & (n = 1 - 2\epsilon^2) \\ 8.6 \times 10^{-62}\ h^{11/9} & (n = 1/2) \\ 3.3 \times 10^{-61}\ h^{6/5} & (n = 0) \\ 1.0 \times 10^{-60}\ h^{13/11} & (n = -1/2) \\ 2.3 \times 10^{-60}\ h^{7/6} & (n = -1). \end{cases} \tag{6.38}$$

We emphasize that these numbers are only order-of-magnitude estimates. Since $\chi \ll 1$ the inflation epoch linear perturbation approximation for the Maxwell field should be valid.

As a further check of the linear perturbation approximation we have also evaluated χ, eqns. (6.25) and (6.26), on a scale ten times the Hubble scale,

$$\frac{a}{k} = \frac{10}{H}, \tag{6.39}$$

at a redshift z,

$$\frac{z}{z_{R\Phi}} \simeq \begin{cases} 4.3\times 10^{22}\epsilon^{1/2}\,h^{-1} & (n=1-2\epsilon^2) \\ 2.2\times 10^{24}\ h^{-10/9} & (n=1/2) \\ 7.3\times 10^{25}\ h^{-6/5} & (n=0) \\ 7.7\times 10^{26}\ h^{-14/11} & (n=-1/2) \\ 9.1\times 10^{27}\ h^{-4/3} & (n=-1) \end{cases} \tag{6.40}$$

[$z_{R\Phi}$ is given in eqn. (6.15)], chosen to ensure that the current perturbation scale $2\pi a_{\rm now}/k = 10^{-3}/H_{\rm now}$. χ ranges between $3.4\times 10^{-9}\epsilon\, h^{-1}$ (for $\alpha r = 4-2\bar\epsilon$, $n = 1-2\epsilon^2$) and $2.4\times 10^{-122}\epsilon^{-3/2}\, h^4$ (for $\alpha r = -6+2\bar\epsilon$, $n = 1-2\epsilon^2$). [We note that these values of χ are multiplied by factors raised to the $\bar\epsilon^{\rm th}$ power, which are of order unity for small enough $\bar\epsilon$.]

A measure of the ratio of the position space Maxwell energy density to the background scalar field energy density is

$$\Upsilon = \frac{k^3}{\pi^2}\frac{\epsilon_I(k,t)}{\rho_{b\Phi}(t)}, \tag{6.41}$$

where the inflation epoch Maxwell energy density ϵ_I is defined in eqn. (2.66) and $\rho_{b\Phi}$ is given in eqn. (2.25). We emphasize that, as discussed in Sec. 2.5, when $\alpha \neq 0$ the large-scale part of the inflation epoch Maxwell energy density increases slower than a^{-4} as a is decreased. During inflation, on large scales, we find from eqns. (2.69) – (2.71), for $\hat\nu > 0$ ($\alpha r > -1$),

$$\begin{aligned}\Upsilon \simeq\, & 2.1\times 10^{-127}\left(\frac{k\, z_{R\Phi}}{aH}\right)^4 \left(\frac{z}{z_{R\Phi}}\right)^{2(1-n)/(3-n)} \\ & \times\left[\Gamma^2\!\left(\frac{1}{2}+\frac{\alpha r}{2}\right)\left(\frac{4}{3-n}\frac{aH}{k}\right)^{\alpha r} + \Gamma^2\!\left(\frac{1}{2}-\frac{\alpha r}{2}\right)\left(\frac{4}{3-n}\frac{aH}{k}\right)^{-\alpha r}\right]\end{aligned} \tag{6.42}$$

(where the first term in the square parentheses is the magnetic contribution and the second term is the electric contribution), and from eqns. (2.73) and (2.71), for $\hat\nu < 0$ ($\alpha r < -1$),

$$\Upsilon \simeq 2.1\times 10^{-127}\left(\frac{k\, z_{R\Phi}}{aH}\right)^4 \left(\frac{z}{z_{R\Phi}}\right)^{2(1-n)/(3-n)} \Gamma^2\!\left(\frac{1}{2}-\frac{\alpha r}{2}\right)\left(\frac{4}{3-n}\frac{aH}{k}\right)^{-\alpha r}, \tag{6.43}$$

which is the electric contribution (the magnetic contribution is suppressed). The small numerical prefactor in these equations is a consequence of $H_{BR}^2/[m_p^2(1+z_{BR})^4] \simeq 7.7\times 10^{-127}$.

Evaluating eqns. (6.42) and (6.43) at reheating ($z = z_{R\Phi}$) and on a scale given by eqn. (6.27), we find, for $\alpha r = 4 - 2\bar{\epsilon}$ or $\alpha r = -4$,

$$\Upsilon \simeq \begin{cases} 5.0 \times 10^{-7} \epsilon^2 & (n = 1 - 2\epsilon^2) \\ 5.0 \times 10^{-20}\, h^{4/9} & (n = 1/2) \\ 2.7 \times 10^{-29}\, h^{4/5} & (n = 0) \\ 4.4 \times 10^{-37}\, h^{12/11} & (n = -1/2) \\ 7.6 \times 10^{-43}\, h^{4/3} & (n = -1) \end{cases} \tag{6.44}$$

[the values of α for this and following equations may be found in eqns. (6.16) – (6.23)]. Once again, for $\alpha r = 4 - 2\bar{\epsilon}$ these numbers must be multiplied by a factor raised to the $\bar{\epsilon}^{\text{th}}$ power, which is of order unity for small enough $\bar{\epsilon}$. For $\alpha r = 7/2$ or $\alpha r = -7/2$,

$$\Upsilon \simeq \begin{cases} 3.9 \times 10^{-19} \epsilon^{7/4}\, h^{1/2} & (n = 1 - 2\epsilon^2) \\ 1.6 \times 10^{-30}\ h^{8/9} & (n = 1/2) \\ 1.3 \times 10^{-38}\ h^{6/5} & (n = 0) \\ 2.0 \times 10^{-45}\ h^{16/11} & (n = -1/2) \\ 1.8 \times 10^{-50}\ h^{5/3} & (n = -1), \end{cases} \tag{6.45}$$

for $\alpha r = 5/2$ or $\alpha r = -5/2$,

$$\Upsilon \simeq \begin{cases} 3.0 \times 10^{-43} \epsilon^{5/4}\, h^{3/2} & (n = 1 - 2\epsilon^2) \\ 2.2 \times 10^{-51}\ h^{16/9} & (n = 1/2) \\ 3.6 \times 10^{-57}\ h^{2} & (n = 0) \\ 4.9 \times 10^{-62}\ h^{24/11} & (n = -1/2) \\ 1.2 \times 10^{-65}\ h^{7/3} & (n = -1), \end{cases} \tag{6.46}$$

for $\alpha r = 2$ or $\alpha r = -2$,

$$\Upsilon \simeq \begin{cases} 3.0 \times 10^{-55} \epsilon\ h^{2} & (n = 1 - 2\epsilon^2) \\ 9.5 \times 10^{-62}\ h^{20/9} & (n = 1/2) \\ 2.2 \times 10^{-66}\ h^{12/5} & (n = 0) \\ 2.8 \times 10^{-70}\ h^{28/11} & (n = -1/2) \\ 3.7 \times 10^{-73}\ h^{8/3} & (n = -1), \end{cases} \tag{6.47}$$

for $\alpha r = 3/2$ or $\alpha r = -3/2$,

$$\Upsilon \simeq \begin{cases} 3.4 \times 10^{-67} \epsilon^{3/4}\, h^{5/2} & (n = 1 - 2\epsilon^2) \\ 4.5 \times 10^{-72}\ h^{8/3} & (n = 1/2) \\ 1.5 \times 10^{-75}\ h^{14/5} & (n = 0) \\ 1.8 \times 10^{-78}\ h^{32/11} & (n = -1/2) \\ 1.3 \times 10^{-80}\ h^{3} & (n = -1), \end{cases} \tag{6.48}$$

for $\alpha r = 1/2$ or $\alpha r = -1/2$,

$$\Upsilon \simeq \begin{cases} 7.2 \times 10^{-91} \epsilon^{1/4}\, h^{7/2} & (n = 1 - 2\epsilon^2) \\ 1.7 \times 10^{-92}\ h^{32/9} & (n = 1/2) \\ 1.2 \times 10^{-93}\ h^{18/5} & (n = 0) \\ 1.3 \times 10^{-94}\ h^{40/11} & (n = -1/2) \\ 2.4 \times 10^{-95}\ h^{11/3} & (n = -1), \end{cases} \tag{6.49}$$

for the 'free' case when $\alpha = 0$,

$$\Upsilon \simeq 3.2 \times 10^{-102}\ h^4, \tag{6.50}$$

for $\alpha r = -9/2$,

$$\Upsilon \simeq \begin{cases} 6.7 \times 10^{5} \epsilon^{9/4}\, h^{-1/2} & (n = 1 - 2\epsilon^2) \\ 1.6 \times 10^{-9} & (n = 1/2) \\ 6.0 \times 10^{-20}\ h^{2/5} & (n = 0) \\ 1.0 \times 10^{-28}\ h^{8/11} & (n = -1/2) \\ 3.4 \times 10^{-35}\ h & (n = -1), \end{cases} \tag{6.51}$$

for $\alpha r = -11/2$,

$$\Upsilon \simeq \begin{cases} 1.5 \times 10^{30} \epsilon^{11/4}\, h^{-3/2} & (n = 1 - 2\epsilon^2) \\ 2.0 \times 10^{12}\ h^{-8/9} & (n = 1/2) \\ 3.5 \times 10^{-1}\ h^{-2/5} & (n = 0) \\ 7.0 \times 10^{-12} & (n = -1/2) \\ 8.2 \times 10^{-20}\ h^{1/3} & (n = -1), \end{cases} \tag{6.52}$$

and for $\alpha r = -6 + 2\bar{\epsilon}$ (where we have suppressed an $\bar{\epsilon}$ dependent factor which is of order unity for small enough $\bar{\epsilon}$),

$$\Upsilon \simeq \begin{cases} 2.3 \times 10^{42} \epsilon^3 \, h^{-2} & (n = 1 - 2\epsilon^2) \\ 7.3 \times 10^{22} \; h^{-4/3} & (n = 1/2) \\ 9.2 \times 10^{8} \; h^{-4/5} & (n = 0) \\ 1.9 \times 10^{-3} \; h^{-4/11} & (n = -1/2) \\ 4.3 \times 10^{-12} & (n = -1). \end{cases} \tag{6.53}$$

We emphasize that these numbers are only order-of-magnitude estimates. We note that in these equations, when $\alpha r \leq -1/2$ it is the electric field energy density that contributes to the large-scale part of Υ, when $\alpha r \geq 1/2$ it is the magnetic field energy density that contributes, and when $\alpha r = 0$ both the electric field and magnetic field terms contribute.

If αr is near $-6 + 2\bar{\epsilon}$ and n is near $1 - 2\epsilon^2$ (but not exactly unity) Υ is larger than unity, eqns. (6.53), (6.52) and (6.51), and the large-scale vector potential energy density is greater than that in the homogeneous part of the inflation epoch scalar field. This region of parameter space must be excluded since the assumption that the scalar field dominates during inflation is violated. For the rest of the allowed parameter space the scalar field does dominate, eqns. (6.44) – (6.50) and parts of eqns. (6.51), (6.52) and (6.53).

As a further check we have evaluated Υ, eqns. (6.42) and (6.43), at the redshift given in eqns. (6.40) and for a perturbation length scale given in eqn. (6.39). Υ ranges between $1.3 \times 10^{-3} \epsilon^2$ (for $\alpha r = -6 + 2\bar{\epsilon}$, $n = 1 - 2\epsilon^2$, and suppressing an $\bar{\epsilon}$ and ϵ dependent factor which is of order unity for small enough $\bar{\epsilon}$ and ϵ) to 2.5×10^{-18} (for $\alpha r = 0$, $n = -1$).

We note, from the discussion below eqn. (2.67), that during inflation the dielectric 'constant' ϵ and magnetic permeability μ are [using eqns. (2.21) – (2.23), (3.45) and (6.11)],

$$\epsilon = \left[\left(\frac{a(t)}{a(t_{R\Phi})} \right)^{2/(3-n)} \right]^{\alpha r} = \mu^{-1}, \tag{6.54}$$

where, during inflation when $a(t) \ll a(t_{R\Phi})$, the factor in the square parentheses is $\ll 1$ (it is smallest for $n = 1 - 2\epsilon^2$ and largest for $n = -1$), and at the end of inflation, when $a(t) = a(t_{R\Phi})$, it is unity. For fixed n and $a(t) < a(t_{R\Phi})$, ϵ and μ^{-1} are smallest when

$\alpha r = 4 - 2\bar{\epsilon}$ and largest when $\alpha r = -6 + 2\bar{\epsilon}$. Since the effective Maxwell 'coupling constant' during inflation is $\propto 1/\sqrt{\epsilon}$, eqn. (2.1), we might call the case when αr is near $4 - 2\bar{\epsilon}$ the 'strong-coupling' case while the case when αr is near $-6 + 2\bar{\epsilon}$ might be called the 'weak-coupling' case; we emphasize that we do not use an expansion in the effective Maxwell 'coupling constant' to derive our results. If there is a problem with the large inflation epoch 'coupling constant' when fermions are added to the model, one might be able to use a different functional form for f, eqn. (2.21), to ensure that the 'coupling constant' only varied when the scales of interest left the Hubble radius (and did not become very large in the past).

7. Conclusion

We have studied the generation and evolution of electromagnetic perturbations in a simple scalar field inflation modified hot big bang cosmological model. The maximum value of the present intergalactic magnetic field, on a scale of one-thousandth the Hubble scale, might be as large as $\sim 10^{-9}$G. Since we have had to make use of a number of simplifying approximations this should be considered a preliminary estimate. (We emphasize that for the larger values of the present magnetic field the inflation epoch effective Maxwell coupling 'constant' is large at early times. In the weak coupling case the maximum field is $\sim 10^{-23}$G.)

There are a number of issues that need to be examined more carefully than we have done here before one can gauge the reliability of our estimate of the magnetic field strength. We have assumed that the effects of charged particles during inflation may be ignored; this needs to be verified. A better model of the reheating transition would be useful. A better way of phenomenologically incorporating the physics of the effective Maxwell coupling constant becoming time-independent after this transition would also be useful. It would seem to be straightforward to improve upon our treatment of the radiation-baryon transition. It would be useful to account for finite conductivity effects during baryon dominance. The effects of the electroweak (and any earlier) transition should be more carefully accounted for. It would seem that an analysis of a more realistic model is called for; this model should, at least at low energies, more closely resemble the standard particle physics model.

We suspect that there are other early universe models which generate intergalactic magnetic fields [besides the obvious ones with different functional forms for $f(\Phi)$].

There now seems to be sufficient theoretical motivation to justify an investigation of the consequences of a significant intergalactic magnetic field.

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Appendix A: 3 + 1 Formulation of Electrodynamics in Curved Spacetime

To be able to discuss the effect of charged particles on the evolution of the Maxwell field and to be able to make contact with the observational upper bound on a large-scale cosmological magnetic field we need to know how to define the electric and magnetic fields in terms of the vector potential. This has been accomplished in Refs. [75-77], and in this Appendix we reproduce, in our notation and without proof, a slightly modified and slightly generalized version of the results of Thorne and Macdonald [77].

We consider fundamental observers with four-velocity u^μ whose worldlines (labelled by the observer's proper time τ) are orthogonal to spatial hypersurfaces (labelled by global time t) which fill spacetime; the four-velocity is normalized so that

$$u_\mu u^\mu = 1. \tag{A1}$$

A four-tensor $M^{\mu\nu\cdots}$ orthogonal to the four-velocity,

$$u_\mu \, M^{\mu\nu\cdots} = 0, \quad u_\nu \, M^{\mu\nu\cdots} = 0, \quad \cdots, \tag{A2}$$

will be called a spatial tensor and will be denoted by $\underline{M}$.

The spatial hypersurface metric is

$$P_{\mu\nu} = g_{\mu\nu} - u_\mu \, u_\nu, \tag{A3}$$

and satisfies

$$u^\mu \, P_{\mu\nu} = 0 = P_{\mu\nu} \, u^\nu, \tag{A4}$$

$$P_\mu{}^\mu = 3, \tag{A5}$$

$$P_\mu{}^\nu \, P_\nu{}^\lambda = P_\mu{}^\lambda, \tag{A6}$$

so $P_{\mu\nu}$ is spatial tensor as well as a projection tensor onto the spatial hypersurface.

The kinematic properties of worldlines of fundamental observers may be classified by their expansion,

$$\theta = u^\mu{}_{;\mu} \tag{A7}$$

(the semicolon denotes a four-dimensional covariant derivative), their four-acceleration,

$$a_\mu = u_{\mu;\nu} \, u^\nu, \tag{A8}$$

their rotation (or vorticity) tensor,

$$\bar{\omega}_{\mu\nu} = \frac{1}{2} \left(u_{\mu;\lambda} \, P^\lambda{}_\nu - u_{\nu;\lambda} \, P^\lambda{}_\mu \right) \tag{A9}$$

(which is antisymmetric in μ and ν), and their shear tensor,

$$\sigma_{\mu\nu} = \frac{1}{2} \left(u_{\mu;\lambda} \, P^\lambda{}_\nu + u_{\nu;\lambda} \, P^\lambda{}_\mu \right) - \frac{1}{3} \, \theta \, P_{\mu\nu} \tag{A10}$$

(which is symmetric in μ and ν and traceless). Using eqns. $(A1)$ and $(A4)$, it is straightforward to establish that a_μ, $\bar{\omega}_{\mu\nu}$ and $\sigma_{\mu\nu}$ are spatial tensors. It may also be shown that

$$u_{\mu;\nu} = \bar{\omega}_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\,\theta\, P_{\mu\nu} + a_\mu\, u_\nu. \qquad (A11)$$

The four-dimensional totally antisymmetric tensor density is denoted by $\epsilon^{\mu\nu\lambda\rho}$ (where $\epsilon^{0123} = 1$) and the corresponding object with lower indices is defined by using

$$\epsilon_\alpha{}^{\nu\lambda\rho} = g^{-1/4} g_{\alpha\mu}\, \epsilon^{\mu\nu\lambda\rho}, \qquad (A12)$$

where g is the determinant of the metric tensor. Completeness may be expressed as

$$\epsilon^{\alpha_1\cdots\alpha_p\gamma_{p+1}\cdots\gamma_4}\, \epsilon_{\beta_1\cdots\beta_p\gamma_{p+1}\cdots\gamma_4} = p!(4-p)!\delta^{\alpha_1}_{[\beta_1} \cdots \delta^{\alpha_p}_{\beta_p]}, \qquad (A13)$$

where p can be 0, 1, 2 or 3 and the square brackets, $_{[\ ]}$, represent antisymmetrization normalized to unit weight. It may be shown that the three-dimensional totally antisymmetric tensor density $\hat{\epsilon}^{\mu\nu\lambda}$ ($\hat{\epsilon}^{123} = -1$) is defined by

$$u_\mu \frac{\epsilon^{\mu\alpha\beta\gamma}}{\sqrt{-g}} = -\frac{\hat{\epsilon}^{\rho\sigma\tau}}{\sqrt{-P}}\, P_\rho{}^\alpha\, P_\sigma{}^\beta\, P_\tau{}^\gamma, \qquad (A14)$$

where P is the three-determinant of the spatial metric tensor. Relations similar to eqns. ($A12$) and ($A13$) hold for $\hat{\epsilon}^{\mu\nu\lambda}$.

The dot and cross products for spatial vectors $\underline{M}$ and $\underline{N}$ are defined by

$$\underline{M} \cdot \underline{N} = P^{\mu\nu}\, M_\mu\, N_\nu, \qquad (A15)$$

$$(\underline{M} \times \underline{N})^\lambda = \frac{\hat{\epsilon}^{\lambda\mu\nu}}{\sqrt{-P}}\, M_\mu\, N_\nu; \qquad (A16)$$

in a locally Minkowskian coordinate system, with $u^i = 0$, eqns. ($A15$) and ($A16$) coincide with the standard definitions upto a minus sign. The covariant spatial gradient $\underline{\nabla}$, of a spatial vector $\underline{M}$, $\underline{\nabla}\,\underline{M}$, is a spatial tensor, with spacetime components

$$M^\mu{}_{|\nu} = P^\mu{}_\alpha\, P_\nu{}^\beta\, M^\alpha{}_{;\beta} \qquad (A17)$$

(where $_|$ represents the covariant spatial derivative), so the divergence and curl of $\underline{M}$ are

$$\underline{\nabla} \cdot \underline{M} = M^\mu{}_{|\mu}, \qquad (A18)$$

$$(\underline{\nabla} \times \underline{M})^\lambda = \frac{\hat{\epsilon}^{\lambda\mu\nu}}{\sqrt{-P}} M_{\nu|\mu}. \tag{A19}$$

Since the spatial hypersurface is not necessarily flat, covariant spatial derivatives do not necessarily commute; furthermore, the divergence and curl do not obey all of the identities familiar from vector calculus on a vorticity-free hypersurface.

The time derivative of a spatial tensor is defined by Fermi-Walker (gyroscope) transport along the worldline of a fundamental observer. For a spatial vector this is

$$D_{(\tau)} M^\mu = P^\mu{}_\nu \, M^\nu{}_{;\alpha} \, u^\alpha , \tag{A20}$$

where $D_{(\tau)}M^\mu$ are the spacetime components of the time derivative of a spatial vector $D_{(\tau)}\underline{M}$. The generalization of eqn. $(A20)$ to arbitrary spatial tensors is straightforward and it may be shown that the time derivative of the spatial metric, $D_{(\tau)}P_{\alpha\beta}$, vanishes.

The spatial vector potential $\underline{A}$, current density vector $\underline{j}$, electric field vector $\underline{E}$, and magnetic field vector $\underline{B}$, have spacetime components orthogonal to the four-velocity,

$$A_\mu \, u^\mu = 0, \tag{A21}$$

$$j_\mu \, u^\mu = 0, \tag{A22}$$

$$E_\mu \, u^\mu = 0, \tag{A23}$$

$$B_\mu \, u^\mu = 0. \tag{A24}$$

The four-dimensional vector potential $\mathcal{A}_\mu$ (not to be confused with the $\mathcal{A}_\mu$ of Sec. 2), the four-dimensional current density vector J_μ, and the Maxwell field-strength tensor $F_{\mu\nu}$, are related to the above fields through

$$\mathcal{A}_\mu = \phi \, u_\mu + A_\mu , \tag{A25}$$

$$J_\mu = \rho_e \, u_\mu - j_\mu , \tag{A26}$$

$$F_{\mu\nu} = E_\mu \, u_\nu - u_\mu \, E_\nu - \sqrt{-g} \, \epsilon_{\mu\nu\lambda\rho} \, u^\lambda \, B^\rho , \tag{A27}$$

where ϕ is the Maxwell scalar potential (not to be confused with the scalar field perturbation of Sec. 2) and ρ_e is the charge density. Clearly,

$$\phi = u^\mu\, \mathcal{A}_\mu, \tag{A28}$$

$$A_\mu = P_\mu{}^\nu\, \mathcal{A}_\nu, \tag{A29}$$

$$\rho_e = u^\mu\, J_\mu, \tag{A30}$$

$$j_\mu = -\, P_\mu{}^\nu\, J_\nu, \tag{A31}$$

$$E_\mu = -\, u^\nu\, F_{\nu\mu}, \tag{A32}$$

$$B^\mu = \frac{1}{2}\frac{\epsilon^{\mu\nu\lambda\rho}}{\sqrt{-g}}\, u_\nu\, F_{\lambda\rho}. \tag{A33}$$

The Maxwell equations are

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu, \tag{A34}$$

$$F_{[\mu\nu;\lambda]} = 0. \tag{A35}$$

The component of eqn. $(A34)$ normal to the spatial hypersurface may be shown to be

$$P^{\mu\nu}\left[-E_{\nu;\mu} - 2\omega_\mu\, B_\nu\right] = 4\pi\rho_e, \tag{A36}$$

where we have defined the vorticity vector ω^μ by

$$\bar{\omega}_{\alpha\beta} = \sqrt{-g}\, \epsilon_{\alpha\beta\gamma\delta}\, u^\gamma \omega^\delta, \tag{A37}$$

$$\omega^\alpha = -\frac{1}{2}\frac{\epsilon^{\alpha\beta\gamma\delta}}{\sqrt{-g}}\, u_\beta\, \bar{\omega}_{\gamma\delta}, \tag{A38}$$

so ω^α is a spatial vector. The component of eqn. $(A34)$ in the spatial hypersurface is

$$D_{(\tau)}E^\gamma - P^{\gamma\alpha}\left[\sigma_{\alpha\beta} - \frac{2}{3}\,\theta\, P_{\alpha\beta}\right] E^\beta + P^\gamma{}_\alpha\, \frac{\hat{\epsilon}^{\alpha\mu\nu}}{\sqrt{-P}}\, P_\mu{}^\beta\, P_\nu{}^\delta\, \omega_\beta\, E_\delta$$
$$= P^\gamma{}_\alpha\, \frac{\hat{\epsilon}^{\alpha\mu\nu}}{\sqrt{-P}}\, P_\mu{}^\beta\, P_\nu{}^\delta\left[B_{\delta;\beta} - B_\delta\, a_\beta\right] - 4\pi P^\gamma{}_\alpha\, j^\alpha. \tag{A39}$$

Similarly, eqn. $(A35)$ may be reexpressed as

$$P^{\mu\nu}\left[B_{\nu;\mu} - 2\omega_\mu\, E_\nu\right] = 0, \tag{A40}$$

$$D_{(\tau)}B^{\gamma} - P^{\gamma\alpha}\left[\sigma_{\alpha\beta} - \frac{2}{3}\,\theta\,P_{\alpha\beta}\right] B^{\beta} + P^{\gamma}{}_{\alpha}\,\frac{\hat{\epsilon}^{\alpha\mu\nu}}{\sqrt{-P}}\,P_{\mu}{}^{\beta}\,P_{\nu}{}^{\delta}\,\omega_{\beta}\,B_{\delta}$$
$$= -P^{\gamma}{}_{\alpha}\,\frac{\hat{\epsilon}^{\alpha\mu\nu}}{\sqrt{-P}}\,P_{\mu}{}^{\beta}\,P_{\nu}{}^{\delta}\left[E_{\delta;\beta} - E_{\delta}\,a_{\beta}\right]. \tag{A41}$$

In three-dimensional notation eqns. $(A36)$ and $(A40)$ are

$$-\underline{\nabla}\cdot\underline{E} - 2\underline{\omega}\cdot\underline{B} = 4\pi\rho_e, \tag{A42}$$

$$\underline{\nabla}\cdot\underline{B} - 2\underline{\omega}\cdot\underline{E} = 0, \tag{A43}$$

while eqns. $(A39)$ and $(A41)$ are

$$D_{(\tau)}\underline{E} + \frac{2}{3}\,\theta\,\underline{E} - \underline{\sigma}\cdot\underline{E} + \underline{\omega}\times\underline{E} = \underline{\nabla}\times\underline{B} - \underline{a}\times\underline{B} - 4\pi\underline{j}, \tag{A44}$$
$$D_{(\tau)}\underline{B} + \frac{2}{3}\,\theta\,\underline{B} - \underline{\sigma}\cdot\underline{B} + \underline{\omega}\times\underline{B} = -\underline{\nabla}\times\underline{E} + \underline{a}\times\underline{E}. \tag{A45}$$

In terms of the four-potential $\mathcal{A}_{\mu}$, eqn. $(A25)$, the Maxwell field-strength tensor is

$$F_{\mu\nu} = \mathcal{A}_{\nu;\mu} - \mathcal{A}_{\mu;\nu}. \tag{A46}$$

Using eqns. $(A32)$ and $(A33)$ we find

$$E_{\mu} = P_{\mu}{}^{\nu}\left[\phi_{;\nu} - a_{\nu}\,\phi - u^{\lambda}\,A_{\nu;\lambda} - A^{\lambda}(\bar{\omega}_{\lambda\nu} + \sigma_{\lambda\nu}) - \frac{1}{3}\,\theta\,A_{\nu}\right], \tag{A47}$$
$$B^{\mu} = -u_{\nu}\,\frac{\epsilon^{\nu\mu\lambda\rho}}{\sqrt{-g}}\,A_{\rho;\lambda} + 2\phi\,\omega^{\mu}, \tag{A48}$$

or in three-dimensional notation

$$\underline{E} = \underline{\nabla}\phi - \underline{a}\,\phi - D_{(\tau)}\underline{A} - \underline{A}\cdot(\underline{\bar{\omega}} + \underline{\sigma}) - \frac{1}{3}\,\theta\underline{A}, \tag{A49}$$

$$\underline{B} = \underline{\nabla}\times\underline{A} + 2\phi\,\underline{\omega}. \tag{A50}$$

Charge conservation

$$J^{\mu}{}_{;\mu} = 0, \tag{A51}$$

may be reexpressed as

$$D_{(\tau)}\rho_e + \theta\rho_e - \underline{\nabla}\cdot\underline{j} + \underline{a}\cdot\underline{j} = 0. \tag{A52}$$

A particle of mass m, charge q and four-velocity

$$W^{\mu} = \frac{1}{\sqrt{1 + w_{\mu} w^{\mu}}}\,(u^{\mu} + w^{\mu}), \tag{A53}$$

where w_i is the particle's velocity in the rest frame of the local fundamental observer, and

$$u_{\mu} w^{\mu} = 0, \tag{A54}$$

obeys the force law

$$W^{\mu}\, W^{\nu}{}_{;\mu} = -\frac{q}{m}\, F^{\nu\mu}\, W_{\mu}. \tag{A55}$$

The particle's momentum is

$$p^{\mu} = \frac{m\, w^{\mu}}{\sqrt{1 + w_{\mu} w^{\mu}}}, \tag{A56}$$

while the energy is given by

$$E = m\, u_{\mu} W^{\mu} = \frac{m}{\sqrt{1 + w_{\mu} w^{\mu}}}. \tag{A57}$$

Projecting eqn. ($A55$) tangentially and normally to the spatial hypersurface we have,

$$(D_{(\tau)} + \underline{w} \cdot \underline{\nabla})\underline{p} = -\frac{m\, \underline{a}}{\sqrt{1 + \underline{w} \cdot \underline{w}}} - (\underline{\bar{\omega}} + \underline{\sigma}) \cdot \underline{p} - \frac{1}{3}\, \theta\, \underline{p} - q\,(\underline{E} - \underline{w} \times \underline{B}), \tag{A58}$$

$$(D_{(\tau)} + \underline{w} \cdot \underline{\nabla})E = \underline{a} \cdot \underline{p} + \underline{w} \cdot \underline{\sigma} \cdot \underline{p} + \frac{1}{3}\, \theta\, \underline{w} \cdot \underline{p} + q\, \underline{E} \cdot \underline{w}; \tag{A59}$$

the last term on the right hand side of eqn. ($A58$) is the Lorentz force term and its form justifies the definition of the electric and magnetic fields used in eqn. ($A27$) (the minus sign is a consequence of our definition of the cross product, see below).

Finally we note that to complete the system of equations ($A42$) – ($A45$) we need, besides the (charged) fluid equations of motion, expressions for the charge and current densities. For our purposes it suffices to assume overall charge neutrality,

$$\rho_e = 0, \qquad (A60)$$

and the simplified (approximate) Ohm's law,

$$\underline{j} = \sigma(\underline{E} - \underline{w} \times \underline{B}), \qquad (A61)$$

where σ is the conductivity (not to be confused with the shear) and $\underline{w}$ is the fluid velocity.

Appendix B: Linearization of 3 + 1 Formulation of Electrodynamics in an Ideal Fluid Dominated FLRW Model

Choosing the fundamental observers to comove with the fluid (so that global time is the observer's proper time) and using the synchronous gauge metric expansion, eqn. (2.7), and fluid coordinate peculiar velocity expansion, eqns. (3.10) and (3.11), the spatial hypersurface metric, to first order in the perturbation, is

$$P_{00} = 0, \qquad (B1)$$

$$P_{0i} = a^2\, u^i, \qquad (B2)$$

$$P_{ij} = -a^2(\delta_{ij} - h_{ij}). \qquad (B3)$$

To first order in the perturbations we also have, for the expansion and the four-acceleration,

$$\theta = 3\,\frac{\dot{a}}{a} + u^i{}_{,i} - \frac{1}{2}\,\dot{h}, \qquad (B4)$$

$$a^0 = 0, \qquad (B5)$$

$$a^i = \dot{u}^i + 2\,\frac{\dot{a}}{a}\,u^i, \qquad (B6)$$

and for the shear tensor and the vorticity tensor,

$$\sigma_{00} = 0 = \sigma_{0i}, \qquad (B7)$$

$$\sigma_{ij} = a^2\left[\frac{1}{3}\,u^k{}_{,k}\,\delta_{ij} - \frac{1}{2}(u^i{}_{,j} + u^j{}_{,i})\right] - a^2\left[\frac{1}{6}\,\dot{h}\,\delta_{ij} - \frac{1}{2}\,\dot{h}_{ij}\right], \qquad (B8)$$

$$\bar{\omega}_{00} = 0 = \bar{\omega}_{0i}, \tag{B9}$$

$$\bar{\omega}_{ij} = -\frac{1}{2}\, a^2 (u^i{}_{,j} - u^j{}_{,i}). \tag{B10}$$

Assuming that the electric and magnetic fields are of first order in the perturbation, we find, from eqns. $(A32)$ and $(A33)$,

$$E_0 = 0, \tag{B11}$$

$$E_i = -F_{0i}, \tag{B12}$$

$$B^0 = 0, \tag{B13}$$

$$B^i = \frac{1}{2a^3}\, \hat{\epsilon}^{ijk}\, F_{jk}. \tag{B14}$$

The Maxwell equations, eqns. $(A42)$ – $(A45)$, reduce to

$$a^{-2}\, \partial_i E_i = 4\pi \rho_e, \tag{B15}$$

$$\partial_i B_i = 0, \tag{B16}$$

$$\dot{E}_i + \frac{\dot{a}}{a}\, E_i = \frac{1}{a}\, \epsilon_{ijk}\, \partial_j B_k - 4\pi j_i, \tag{B17}$$

$$\dot{B}_i + \frac{\dot{a}}{a}\, B_i = -\frac{1}{a}\, \epsilon_{ijk}\, \partial_j E_k, \tag{B18}$$

where we have taken ρ_e and j_i to be of first order in the perturbation and defined $\epsilon_{ijk} = -\hat{\epsilon}_{ijk}$ (so $\epsilon_{123} = 1$). In temporal gauge ($\phi = 0$), in terms of the vector potential perturbation,

$$E_i = -\dot{A}_i, \tag{B19}$$

$$B_i = a^{-1}\, \epsilon_{ijk}\, \partial_j A_k. \tag{B20}$$

Eqn. $(B20)$ reduces eqn. $(B16)$ to an identity while eqns. $(B19)$ and $(B20)$ imply that eqn. $(B18)$ is an identity. (We emphasize that A_i, E_i and B_i are the coordinate vector potential, electric field and magnetic field and, as shown below, the proper vector potential, electric field and magnetic field are A_i/a, E_i/a and B_i/a.) From eqns. $(B15)$ and $(B17)$ or from the linearized version of eqn. $(A52)$ we find that charge conservation is described by

$$\dot{\rho}_e + 3\,\frac{\dot{a}}{a}\,\rho_e + \frac{1}{a^2}\,\partial_i j_i = 0. \tag{B21}$$

Taking the charged particle's velocity, momentum and energy spatial gradient to be of first order in the perturbation, we find that the force law, eqns. $(A58)$ and $(A59)$, becomes

$$\frac{d}{dt}(a^2 p^i) = -m\frac{d}{dt}(a^2 u^i) + q\left[E_i + a\,\epsilon_{ijk} w^j B_k\right], \tag{B22}$$

$$\dot{E} = q\,E_i\,w^i; \tag{B23}$$

the second term in the square parentheses on the right hand side of eqn. $(B22)$ and the term on the right hand side of eqn. $(B23)$ are of second order in the perturbation. For future reference we note that the observer's proper peculiar velocity is

$$v^i = a u^i. \tag{B24}$$

The linearized version of Ohm's law, eqn. $(A61)$, is given by

$$j_i = \sigma\left[E_i + a\,\epsilon_{ijk} w^j B_k\right]; \tag{B25}$$

again, the second term is of higher order.

From eqns. $(A60)$, $(B15)$ – $(B18)$ and $(B25)$ [where we ignore the second term on the right hand side], the equations of motion we shall study are given by

$$\partial_i E_i = 0, \tag{B26}$$

$$\partial_i B_i = 0, \tag{B27}$$

$$\frac{d}{dt}(aE_i) = \epsilon_{ijk}\,\partial_j B_k - 4\pi\sigma(aE_i), \tag{B28}$$

$$\frac{d}{dt}(aB_i) = -\epsilon_{ijk}\,\partial_j E_k; \tag{B29}$$

the corresponding wave equations are

$$\ddot{\mathcal{E}}_i + \left(\frac{\dot{a}}{a} + 4\pi\sigma\right)\dot{\mathcal{E}}_i + 4\pi\sigma\,\frac{\dot{a}}{a}\,\mathcal{E}_i - \frac{1}{a^2}\,\partial_j\partial_j\mathcal{E}_i = 0, \tag{B30}$$

$$\ddot{\mathcal{B}}_i + \left(\frac{\dot{a}}{a} + 4\pi\sigma\right)\dot{\mathcal{B}}_i - \frac{1}{a^2}\,\partial_j\partial_j\mathcal{B}_i = 0, \tag{B31}$$

where we have defined

$$\mathcal{E}_i = aE_i, \tag{B32}$$

$$\mathcal{B}_i = aB_i. \tag{B33}$$

We note that eqns. $(B19)$ and $(B20)$ may be used to reexpress eqns. $(B26)$ and $(B28)$ as equations of motion for the vector potential perturbation,

$$\partial_i\dot{A}_i = 0, \tag{B34}$$

$$\ddot{A}_i + \left(\frac{\dot{a}}{a} + 4\pi\sigma\right)\dot{A}_i + \frac{1}{a^2}\left(\partial_i\partial_j A_j - \partial_j\partial_j A_i\right) = 0. \tag{B35}$$

Comparing to the vector potential equations in the inflation epoch, eqns. (2.18) and (2.19), we see that with the coupling between the scalar field and vector potential used in eqn. (2.1), $\dot{\Phi}_b f'(\Phi_b)/f(\Phi_b)$ plays the role of the conductivity. More precisely, if we replace $4\pi\sigma$ in eqn. $(B28)$ by $\dot{\Phi}_b f'(\Phi_b)/f(\Phi_b)$, the linear perturbation equations $(B26)$ – $(B29)$ are equivalent to the vector potential equations in the general scalar field dominated inflation model of Sec. 2. The electric and magnetic field wave equations for this model agree with eqns. $(B30)$ and $(B31)$ provided we replace the $4\pi\sigma\dot{\mathcal{E}}_i$ term in eqn. $(B30)$ by the time derivative of $4\pi\sigma\mathcal{E}_i$ (since σ is now time dependent).

Introducing the particle's proper velocity

$$W^i = aw^i \tag{B36}$$

(W^i should not be confused with W^μ of Appendix A), and using eqn. $(B24)$, we may rewrite eqns. $(B22)$ and $(B23)$,

$$\frac{m}{a}\frac{d}{dt}(aW^i) = -\frac{m}{a}\frac{d}{dt}(av^i) + q\left[\frac{E_i}{a} + \epsilon_{ijk}W^j\frac{B_k}{a}\right], \tag{B37}$$

$$\dot{E} = q\frac{E_i}{a}W^i; \tag{B38}$$

clearly the proper electric and magnetic fields are E_i/a and B_i/a. We note that eqns. ($B19$) and ($B20$) may be rewritten as

$$\frac{E_i}{a} = -\frac{1}{a}\frac{d}{dt}\left[a\left(\frac{A_i}{a}\right)\right], \tag{B39}$$

$$\frac{B_i}{a} = \epsilon_{ijk}\left(\frac{\partial_j}{a}\right)\left(\frac{A_k}{a}\right), \tag{B40}$$

so A_i/a is the proper vector potential.

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